Exponential Smoothing Methods
Chapter Topics

- Introduction to exponential smoothing
- Simple Exponential Smoothing
- Holt’s Trend Corrected Exponential Smoothing
- Holt-Winters Methods
  - Multiplicative Holt-Winters method
  - Additive Holt-Winters method
Motivation of Exponential Smoothing

- Simple moving average method assigns equal weights \((1/k)\) to all \(k\) data points.
- Arguably, recent observations provide more relevant information than do observations in the past.
- So we want a weighting scheme that assigns decreasing weights to the more distant observations.
Exponential Smoothing

• Exponential smoothing methods give larger weights to more recent observations, and the weights decrease exponentially as the observations become more distant.

• These methods are most effective when the parameters describing the time series are changing SLOWLY over time.
Data vs Methods

- No trend or seasonal pattern?
  - Yes: Single Exponential Smoothing Method
  - No: Linear trend and no seasonal pattern?
    - Yes: Holt’s Trend Corrected Exponential Smoothing Method
    - No: Both trend and seasonal pattern?
      - Yes: Holt-Winters Methods
      - No: Use Other Methods
Simple Exponential Smoothing

• The Simple Exponential Smoothing method is used for forecasting a time series when there is no trend or seasonal pattern, but the mean (or level) of the time series $y_t$ is slowly changing over time.

• NO TREND model

\[ y_t = \beta_o + \varepsilon_t \]
Procedures of Simple Exponential Smoothing Method

• **Step 1:** Compute the initial estimate of the mean (or level) of the series at time period $t = 0$

$$\ell_0 = \bar{y} = \frac{\sum_{t=1}^{n} y_t}{n}$$

• **Step 2:** Compute the updated estimate by using the smoothing equation

$$\ell_T = \alpha y_T + (1-\alpha)\ell_{T-1}$$

where $\alpha$ is a smoothing constant between 0 and 1.
Procedures of Simple Exponential Smoothing Method

Note that

\[ \ell_T = \alpha y_T + (1 - \alpha)\ell_{T-1} \]

\[ = \alpha y_T + (1 - \alpha)[\alpha y_{T-1} + (1 - \alpha)\ell_{T-2}] \]

\[ = \alpha y_T + (1 - \alpha)\alpha y_{T-1} + (1 - \alpha)^2\ell_{T-2} \]

\[ = \alpha y_T + (1 - \alpha)\alpha y_{T-1} + (1 - \alpha)^2 \alpha y_{T-2} + ... + (1 - \alpha)^{T-1} \alpha y_1 + (1 - \alpha)^T \ell_0 \]

The coefficients measuring the contributions of the observations decrease exponentially over time.
Simple Exponential Smoothing

- Point forecast made at time $T$ for $y_{T+p}$
  \[ \hat{y}_{T+p}(T) = \ell_T \quad (p = 1, 2, 3, \ldots) \]

- SSE, MSE, and the standard errors at time $T$
  \[
  SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2 \\
  MSE = \frac{SSE}{T - 1}, \quad s = \sqrt{MSE}
  \]

Note: There is no theoretical justification for dividing SSE by ($T$ – number of smoothing constants). However, we use this divisor because it agrees to the computation of $s$ in Box-Jenkins models introduced later.
Example: Cod Catch

- The Bay City Seafood Company recorded the monthly cod catch for the previous two years, as given below.

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<th>Year 2</th>
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<td>December</td>
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Example: Cod Catch

- The plot of these data suggests that there is no trend or seasonal pattern. Therefore, a NO TREND model is suggested: \( y_t = \beta_o + \varepsilon_t \)

It is also possible that the mean (or level) is slowly changing over time.
Example: Cod Catch

- **Step 1**: Compute $\ell_0$ by averaging the first twelve time series values.

$$
\ell_0 = \frac{\sum_{t=1}^{12} y_t}{12} = \frac{362 + 381 + \ldots + 343}{12} = 360.6667
$$

Though there is no theoretical justification, it is a common practice to calculate initial estimates of exponential smoothing procedures by using HALF of the historical data.
Example: Cod Catch

• **Step 2:** Begin with the initial estimate $\ell_0 = 360.6667$ and update it by applying the smoothing equation to the 24 observed cod catches.

Set $\alpha = 0.1$ arbitrarily and judge the appropriateness of this choice of $\alpha$ by the model’s in-sample fit.

\[
\ell_1 = \alpha y_1 + (1 - \alpha) \ell_0 = 0.1(362) + 0.9(360.6667) = 360.8000
\]

\[
\ell_2 = \alpha y_2 + (1 - \alpha) \ell_1 = 0.1(381) + 0.9(360.8000) = 362.8200
\]
One-period-ahead Forecasting

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Example: Cod Catch

- Results associated with different values of $\alpha$

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Example: Cod Catch

• Step 3: Find a good value of $\alpha$ that provides the minimum value for MSE (or SSE).
  – Use Solver in Excel as an illustration
Example: Cod Catch

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<th>B</th>
<th>C</th>
<th>D</th>
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Slide 17
Holt’s Trend Corrected Exponential Smoothing

- If a time series is increasing or decreasing approximately at a fixed rate, then it may be described by the LINEAR TREND model

\[ y_t = \beta_0 + \beta_1 t + \varepsilon_t \]

If the values of the parameters \( \beta_0 \) and \( \beta_1 \) are slowly changing over time, Holt’s trend corrected exponential smoothing method can be applied to the time series observations.

**Note:** When neither \( \beta_0 \) nor \( \beta_1 \) is changing over time, regression can be used to forecast future values of \( y_t \).

- Level (or mean) at time \( T \): \( \beta_0 + \beta_1 T \)
- Growth rate (or trend): \( \beta_1 \)
Holt’s Trend Corrected Exponential Smoothing

- A smoothing approach for forecasting such a time series that employs two smoothing constants, denoted by $\alpha$ and $\gamma$.

- There are two estimates $\ell_{T-1}$ and $b_{T-1}$.
  - $\ell_{T-1}$ is the estimate of the level of the time series constructed in time period $T-1$ (This is usually called the permanent component).
  - $b_{T-1}$ is the estimate of the growth rate of the time series constructed in time period $T-1$ (This is usually called the trend component).
Holt’s Trend Corrected Exponential Smoothing

- Level estimate
  \[ \ell_T = \alpha y_T + (1 - \alpha)(\ell_{T-1} + b_{T-1}) \]

- Trend estimate
  \[ b_T = \gamma (\ell_T - \ell_{T-1}) + (1 - \gamma)b_{T-1} \]

where \( \alpha \) = smoothing constant for the level \( (0 \leq \alpha \leq 1) \)

\( \gamma \) = smoothing constant for the trend \( (0 \leq \gamma \leq 1) \)
Holt’s Trend Corrected Exponential Smoothing

• Point forecast made at time $T$ for $y_{T+p}$

$$\hat{y}_{T+p}(T) = \ell_T + pb_T \quad (p = 1, 2, 3, \ldots)$$

• MSE and the standard error $s$ at time $T$

$$SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2$$

$$MSE = \frac{SSE}{T-2}, \quad s = \sqrt{MSE}$$
\[
Y_t = \beta_0 + \beta_1 t + \varepsilon_t
\]

\[
\ell_{T+1} + b_{T+1} = \hat{Y}_{T+2}(T+1)
\]

\[
\gamma(\ell_{T+1} - \ell_T) + (1 - \gamma)b_T = b_{T+1}
\]

\[
\hat{Y}_T(T-1) = \ell_{T-1} + b_{T-1}
\]

\[
\hat{Y}_{T+1}(T) = \ell_T + b_T
\]

\[
\gamma(\ell_T - \ell_{T-1}) + (1 - \gamma)b_{T-1} = b_T
\]
Procedures of Holt’s Trend Corrected Exponential Smoothing

- Use the example of Thermostat Sales as an illustration
Procedures of Holt’s Trend Corrected Exponential Smoothing

• Findings:
  – Overall an upward trend
  – The growth rate has been changing over the 52-week period
  – There is no seasonal pattern
  ⇒ Holt’s trend corrected exponential smoothing method can be applied
Procedures of Holt’s Trend Corrected Exponential Smoothing

• **Step 1**: Obtain initial estimates $\ell_0$ and $b_0$ by fitting a least squares trend line to HALF of the historical data.
  
  - $y$-intercept = $\ell_0$; slope = $b_0$
Procedures of Holt’s Trend Corrected Exponential Smoothing

- **Example**
  - Fit a least squares trend line to the first 26 observations
  - **Trend line**
    \[ \hat{y}_t = 202.6246 - 0.3682t \]
  - \( \ell_0 = 202.6246; \ b_0 = -0.3682 \)
Procedures of Holt’s Trend Corrected Exponential Smoothing

• **Step 2:** Calculate a point forecast of $y_1$ from time 0

  \[ \hat{y}_{T+p}(T) = \ell_T + pb_T \quad T = 0, \ p = 1 \]

• **Example**

  \[ \hat{y}_1(0) = \ell_0 + b_0 = 202.6246 - 0.3682 = 202.2564 \]
Procedures of Holt’s Trend Corrected Exponential Smoothing

- **Step 3**: Update the estimates $\ell_T$ and $b_T$ by using some predetermined values of smoothing constants.
- **Example**: let $\alpha = 0.2$ and $\gamma = 0.1$

\[
\begin{align*}
\ell_1 &= \alpha y_1 + (1 - \alpha)(\ell_0 + b_0) \\
&= 0.2(206) + 0.8(202.6246 - 0.3682) = 203.0051
\end{align*}
\]

\[
\begin{align*}
b_1 &= \gamma(\ell_1 - \ell_0) + (1 - \gamma)b_0 \\
&= 0.1(203.0051 - 202.6246) + 0.9(-0.3682) = -0.2933
\end{align*}
\]

\[
\hat{y}_2(1) = \ell_1 + b_1 = 203.0051 - 0.2933 = 202.7118
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52 45 255 280.9500 4.4428 287.4375 -32.4375 1052.1900
53 46 312 290.7142 4.9749 285.3928 26.6072 707.9453
54 47 296 295.7513 4.9811 295.6891 0.3109 0.0966
55 48 307 301.9860 5.1065 300.7324 6.2676 39.2823
56 49 281 301.8740 4.5846 307.0924 -26.0924 680.8155
57 50 308 306.7669 4.6155 306.4586 1.5414 2.3759
58 51 280 305.1059 3.9878 311.3823 -31.3823 984.8515
59 52 345 316.2750 4.7059 309.0937 35.9063 1289.2627
Procedures of Holt’s Trend Corrected Exponential Smoothing

- **Step 4**: Find the best combination of $\alpha$ and $\gamma$ that minimizes SSE (or MSE)
- Example: Use Solver in Excel as an illustration
<table>
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<th>alpha</th>
<th>gamma</th>
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<td>4.5040</td>
<td>306.4237</td>
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</table>
Holt’s Trend Corrected Exponential Smoothing

- $p$-step-ahead forecast made at time $T$
  \[ \hat{y}_{T+p}(T) = \ell_T + pb_T \quad (p = 1, 2, 3, \ldots) \]

- Example
  - In period 52, the one-period-ahead sales forecast for period 53 is
    \[ \hat{y}_{53}(52) = \ell_{52} + b_{52} = 315.9460 + 4.5040 = 320.45 \]
  - In period 52, the three-period-ahead sales forecast for period 55 is
    \[ \hat{y}_{55}(52) = \ell_{52} + 3b_{52} = 315.9460 + 3(4.5040) = 329.458 \]
Holt’s Trend Corrected Exponential Smoothing

• Example

  - If we observe \( y_{53} = 330 \), we can either find a new set of (optimal) \( \alpha \) and \( \gamma \) that minimize the SSE for 53 periods, or

  - we can simply revise the estimate for the level and growth rate and recalculate the forecasts as follows:

\[
\ell_{53} = \alpha y_{53} + (1-\alpha)(\ell_{52} + b_{52})
\]

\[
= 0.247(330) + 0.753(315.946 + 4.5040) = 322.8089
\]

\[
b_{53} = \gamma (\ell_{53} - \ell_{52}) + (1-\gamma)b_{52}
\]

\[
= 0.095(322.8089 - 315.9460) + 0.905(4.5040) = 4.7281
\]

\[
\hat{y}_{54}(53) = \ell_{53} + b_{53} = 322.8089 + 4.7281 = 327.537
\]

\[
\hat{y}_{55}(53) = \ell_{53} + 2b_{53} = 322.8089 + 2(4.7281) = 332.2651
\]
Holt-Winters Methods

• Two Holt-Winters methods are designed for time series that exhibit linear trend
  - Additive Holt-Winters method: used for time series with constant (additive) seasonal variations
  - Multiplicative Holt-Winters method: used for time series with increasing (multiplicative) seasonal variations
• Holt-Winters method is an exponential smoothing approach for handling SEASONAL data.
• The multiplicative Holt-Winters method is the better known of the two methods.
Multiplicative Holt-Winters Method

- It is generally considered to be best suited to forecasting time series that can be described by the equation:
  \[ y_t = (\beta_0 + \beta_1 t) \times SN_t \times IR_t \]
  - \( SN_t \): seasonal pattern
  - \( IR_t \): irregular component

- This method is appropriate when a time series has a linear trend with a multiplicative seasonal pattern for which the level \((\beta_0 + \beta_1 t)\), growth rate \((\beta_1)\), and the seasonal pattern \((SN_t)\) may be slowly changing over time.
Multiplicative Holt-Winters Method

• Estimate of the level

\[ \ell_T = \alpha(y_T / sn_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1}) \]

• Estimate of the growth rate (or trend)

\[ b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1} \]

• Estimate of the seasonal factor

\[ sn_T = \delta(y_T / \ell_T) + (1-\delta)sn_{T-L} \]

where \( \alpha, \gamma, \) and \( \delta \) are smoothing constants between 0 and 1,

\( L = \) number of seasons in a year (\( L = 12 \) for monthly data, and \( L = 4 \) for quarterly data)
Multiplicative Holt-Winters Method

• Point forecast made at time $T$ for $y_{T+p}$

$$\hat{y}_{T+p}(T) = (\ell_T + pb_T)sn_{T+p-L} \quad (p = 1, 2, 3, \ldots)$$

• MSE and the standard errors at time $T$

$$SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2$$

$$MSE = \frac{SSE}{T - 3}, \quad s = \sqrt{MSE}$$
Procedures of Multiplicative Holt-Winters Method

- Use the Sports Drink example as an illustration

### Quarterly sales of Tiger Sports Drink

<table>
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<tr>
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<th>3</th>
<th>4</th>
<th>5</th>
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<td>120</td>
<td>123</td>
<td>134</td>
<td>142</td>
<td>149</td>
</tr>
</tbody>
</table>

![Graph showing quarterly sales of Tiger Sports Drink]

*Slide 38*
Procedures of Multiplicative Holt-Winters Method

• Observations:
  - Linear upward trend over the 8-year period
  - Magnitude of the seasonal span increases as the level of the time series increases
  \[ \Rightarrow \text{Multiplicative Holt-Winters method can be applied to forecast future sales} \]
Procedures of Multiplicative Holt-Winters Method

• **Step 1:** Obtain initial values for the level \( \ell_0 \), the growth rate \( b_0 \), and the seasonal factors \( sn_{-3}, sn_{-2}, sn_{-1}, \) and \( sn_0 \), by fitting a least squares trend line to at least four or five years of the historical data.

  - \( y \)-intercept = \( \ell_0 \); slope = \( b_0 \)
Procedures of Multiplicative Holt-Winters Method

• Example
  – Fit a least squares trend line to the first 16 observations
  – Trend line
    \[ \hat{y}_t = 95.2500 + 2.4706t \]
  – \( \ell_0 = 95.2500; b_0 = 2.4706 \)

Summary Output

- Regression Statistics
  - Multiple R: 0.403809754
  - R Square: 0.163062318
  - Adjusted R Square: 0.103281055
  - Standard Error: 27.58325823
  - Observations: 16

- ANOVA
  - Regression: 1
  - Residual: 14
  - Total: 15

- Coefficients
  - Intercept: 95.25
  - X Variable 1: 2.470588235
Step 2: Find the initial seasonal factors

1. Compute $\hat{y}_t$ for the in-sample observations used for fitting the regression. In this example, $t = 1, 2, \ldots, 16$.

\[
\begin{align*}
\hat{y}_1 &= 95.2500 + 2.4706(1) = 97.7206 \\
\hat{y}_2 &= 95.2500 + 2.4706(2) = 100.1912 \\
\vdots \\
\hat{y}_{16} &= 95.2500 + 2.4706(16) = 134.7794
\end{align*}
\]
Procedures of Multiplicative Holt-Winters Method

• **Step 2:** Find the initial seasonal factors

2. Detrend the data by computing \( S_t = y_t / \hat{y}_t \) for each time period that is used in finding the least squares regression equation. In this example, \( t = 1, 2, \ldots, 16 \).

\[
\begin{align*}
S_1 &= y_1 / \hat{y}_1 = 72 / 97.7206 = 0.7368 \\
S_2 &= y_2 / \hat{y}_2 = 116 / 100.1912 = 1.1578 \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors

3. Compute the average seasonal values for each of the \( L \) seasons. The \( L \) averages are found by computing the average of the detrended values for the corresponding season. For example, for quarter 1,

\[
\bar{S}_{[1]} = \frac{S_1 + S_5 + S_9 + S_{13}}{4} = \frac{0.7368 + 0.7156 + 0.6894 + 0.6831}{4} = 0.7062
\]
Procedures of Multiplicative Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  4. Multiply the average seasonal values by the normalizing constant

\[
CF = \frac{L}{\sum_{i=1}^{L} S[i]}
\]

such that the average of the seasonal factors is 1. The initial seasonal factors are

\[
sn_{i-L} = S[i](CF) \quad (i = 1, 2, \ldots, L)
\]
Procedures of Multiplicative Holt-Winters Method

• **Step 2**: Find the initial seasonal factors

4. Multiply the average seasonal values by the normalizing constant such that the average of the seasonal factors is 1.

• Example

\[ CF = \frac{4}{3.9999} = 1.0000 \]

\[ s_{n-3} = s_{n-4} = \frac{S_{[1]}(CF)}{} = 0.7062(1) = 0.7062 \]

\[ s_{n-2} = s_{n-4} = \frac{S_{[2]}(CF)}{} = 1.1114(1) = 1.1114 \]

\[ s_{n-1} = s_{n-4} = \frac{S_{[3]}(CF)}{} = 1.2937(1) = 1.2937 \]

\[ s_{n-0} = s_{n-4} = \frac{S_{[1]}(CF)}{} = 0.8886(1) = 0.8886 \]
Procedures of Multiplicative Holt-Winters Method

• **Step 3**: Calculate a point forecast of $y_1$ from time 0 using the initial values

\[
\hat{y}_{T+p}(T) = (\ell_T + pb_T)sn_{T+p-L} \quad (T = 0, \ p = 1)
\]

\[
\hat{y}_1(0) = (\ell_0 + b_0)sn_{1-4} = (\ell_0 + b_0)sn_{-3}
\]

\[
= (95.2500 + 2.4706)(0.7062)
\]

\[
= 69.0103
\]
Procedures of Multiplicative Holt-Winters Method

- **Step 4:** Update the estimates $\ell_T$, $b_T$, and $sn_T$ by using some predetermined values of smoothing constants.
- **Example:** let $\alpha = 0.2$, $\gamma = 0.1$, and $\delta = 0.1$

\[
\ell_1 = \alpha(y_1 / sn_{1-4}) + (1 - \alpha)(\ell_0 + b_0)
= 0.2(72 / 0.7062) + 0.8(95.2500 + 2.4706) = 98.5673
\]

\[
b_1 = \gamma(\ell_1 - \ell_0) + (1 - \gamma)b_0
= 0.1(98.5673 - 95.2500) + 0.9(2.4706) = 2.5553
\]

\[
qn_1 = \delta(y_1 / \ell_1) + (1 - \delta)sn_{1-4}
= 0.1(72 / 98.5673) + 0.9(0.7062) = 0.7086
\]

\[
\hat{y}_2(1) = (\ell_1 + b_1)sn_{2-4}
= (98.5673 + 2.5553)(1.1114) = 112.3876
\]
\[ \ell_2 = \alpha \left( \frac{y_2}{s_{n_{2-4}}} \right) + (1 - \alpha) \left( \ell_1 + b_1 \right) \]
\[ = 0.2 \left( \frac{116}{1.1114} \right) + 0.8 \left( 98.5673 + 2.5553 \right) \]
\[ = 101.7727 \]

\[ b_2 = \gamma (\ell_2 - \ell_1) + (1 - \gamma) b_1 \]
\[ = 0.1 (101.7727 - 98.5673) + 0.9 (2.5553) \]
\[ = 2.62031 \]

\[ s_{n_2} = \delta \left( \frac{y_2}{\ell_2} \right) + (1 - \delta) s_{n_{2-4}} \]
\[ = 0.1 \left( \frac{116}{101.7727} \right) + 0.9 (1.1114) \]
\[ = 1.114239 \]

\[ \hat{y}_3(2) = (\ell_2 + b_2)s_{n_{3-4}} \]
\[ = (101.7727 + 2.62031)(1.2937) \]
\[ = 135.053 \]
\[
\ell_4 = \alpha \left( y_4 / sn_{4-4} \right) + (1 - \alpha) \left( \ell_3 + b_3 \right)
\]
\[= 0.2 \left( \frac{96}{0.8886} \right) + 0.8 \left( 104.5393 + 2.6349 \right) = 107.3464
\]

\[
b_4 = \gamma \left( \ell_4 - \ell_3 \right) + (1 - \gamma) b_3
\]
\[= 0.1 \left( 107.3464 - 104.5393 \right) + 0.9 \left( 2.6349 \right) = 2.65212
\]

\[
\text{sn}_4 = \delta \left( y_4 / \ell_4 \right) + (1 - \delta) \text{sn}_{4-4}
\]
\[= 0.1 \left( 96 / 107.3464 \right) + 0.9 \left( 0.8886 \right) = 0.889170
\]

\[
\hat{y}_5(4) = \left( \ell_4 + b_4 \right) \text{sn}_{5-4}
\]
\[= 107.3464 + 2.65212 \left( 0.7086 \right) = 77.945
\]
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......

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Slide 51
Procedures of Multiplicative Holt-Winters Method

- **Step 5**: Find the most suitable combination of $\alpha$, $\gamma$, and $\delta$ that minimizes SSE (or MSE)
- Example: Use Solver in Excel as an illustration
<table>
<thead>
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<th>Time</th>
<th>y</th>
<th>Level</th>
<th>Growth Rate</th>
<th>Seasonal Factor</th>
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<td>0.8968</td>
<td>150.3283</td>
<td>-1.3283</td>
</tr>
</tbody>
</table>
Multiplicative Holt-Winters Method

- **$p$-step-ahead forecast made at time $T$**

\[
\hat{y}_{T+p}(T) = (\ell_T + pb_T)sn_{T+p-L} \quad (p=1,2,3,...)
\]

- **Example**

\[
\begin{align*}
\hat{y}_{33}(32) &= (\ell_{32} + b_{32})sn_{33-4} = (168.1213 + 2.3028)(0.7044) = 120.0467 \\
\hat{y}_{34}(32) &= (\ell_{32} + 2b_{32})sn_{34-4} = [168.1213 + 2(2.3028)](1.1038) = 190.6560 \\
\hat{y}_{35}(32) &= (\ell_{32} + 3b_{32})sn_{35-4} = [(168.1213 + 3(2.3028)](1.2934) = 226.3834 \\
\hat{y}_{36}(32) &= (\ell_{32} + 4b_{32})sn_{36-4} = [(168.1213 + 4(2.3028)](0.8908) = 157.9678
\end{align*}
\]
Multiplicative Holt-Winters Method

- Example

**Forecast Plot for Sports Drink Sales**

- Observed values
- Forecasts
Additive Holt-Winters Method

- It is generally considered to be best suited to forecasting a time series that can be described by the equation:
  \[ y_t = (\beta_0 + \beta_1 t) + SN_t + IR_t \]
  - \( SN_t \): seasonal pattern
  - \( IR_t \): irregular component

- This method is appropriate when a time series has a linear trend with a constant (additive) seasonal pattern such that the level \((\beta_0 + \beta_1 t)\), growth rate \((\beta_1)\), and the seasonal pattern \((SN_t)\) may be slowly changing over time.
Additive Holt-Winters Method

- Estimate of the level
  \[ \ell_T = \alpha(y_T - sn_{T-L}) + (1 - \alpha)(\ell_{T-1} + b_{T-1}) \]

- Estimate of the growth rate (or trend)
  \[ b_T = \gamma(\ell_T - \ell_{T-1}) + (1 - \gamma)b_{T-1} \]

- Estimate of the seasonal factor
  \[ sn_T = \delta(y_T - \ell_T) + (1 - \delta)sn_{T-L} \]

where \( \alpha, \gamma, \) and \( \delta \) are smoothing constants between 0 and 1,
\( L = \) number of seasons in a year (\( L = 12 \) for monthly data,
and \( L = 4 \) for quarterly data)
Additive Holt-Winters Method

- Point forecast made at time $T$ for $y_{T+p}$

$$\hat{y}_{T+p}(T) = \ell_T + pb_T + sn_{T+p-L} \quad (p = 1, 2, 3,...)$$

- MSE and the standard error $s$ at time $T$

$$SSE = \sum_{t=1}^{T} [y_t - \hat{y}_t(t-1)]^2$$

$$MSE = \frac{SSE}{T-3}, \quad s = \sqrt{MSE}$$
Procedures of Additive Holt-Winters Method

- Consider the Mountain Bike example,

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<tr>
<th>Quarter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<td>11</td>
<td>14</td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>31</td>
<td>33</td>
<td>36</td>
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</tr>
<tr>
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<td>50</td>
<td>55</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>17</td>
<td>21</td>
<td>25</td>
</tr>
</tbody>
</table>
Procedures of Additive Holt-Winters Method

• Observations:
  - Linear upward trend over the 4-year period
  - Magnitude of seasonal span is almost constant as the level of the time series increases

⇒ Additive Holt-Winters method can be applied to forecast future sales
Procedures of Additive Holt-Winters Method

- **Step 1**: Obtain initial values for the level $\ell_0$, the growth rate $b_0$, and the seasonal factors $s_{n-3}$, $s_{n-2}$, $s_{n-1}$, and $s_{n0}$, by fitting a least squares trend line to at least four or five years of the historical data.

  - $y$-intercept = $\ell_0$; slope = $b_0$
Procedures of Additive Holt-Winters Method

- Example
  - Fit a least squares trend line to all 16 observations
  - Trend line
    \[ \hat{y}_t = 20.85 + 0.980882 \ t \]
  - \( \ell_0 = 20.85; \ b_0 = 0.9809 \)
Procedures of Additive Holt-Winters Method

- **Step 2**: Find the initial seasonal factors
  1. Compute $\hat{y}_t$ for each time period that is used in finding the least squares regression equation. In this example, $t = 1, 2, ..., 16$.

\[
\begin{align*}
\hat{y}_1 &= 20.85 + 0.980882(1) = 21.8309 \\
\hat{y}_2 &= 20.85 + 0.980882(2) = 22.8118 \\
\end{align*}
\]

......

\[
\begin{align*}
\hat{y}_{16} &= 20.85 + 0.980882(16) = 36.5441 \\
\end{align*}
\]
**Procedures of Additive Holt-Winters Method**

- **Step 2**: Find the initial seasonal factors

2. Detrend the data by computing $S_t = y_t - \hat{y}_t$ for each observation used in the least squares fit. In this example, $t = 1, 2, \ldots, 16$.

\[ S_1 = y_1 - \hat{y}_1 = 10 - 21.8309 = -11.8309 \]
\[ S_2 = y_2 - \hat{y}_2 = 31 - 22.8112 = 8.1882 \]

......

\[ S_{16} = y_{16} - \hat{y}_{16} = 25 - 36.5441 = -11.5441 \]
Procedures of Additive Holt-Winters Method

- **Step 2:** Find the initial seasonal factors

3. Compute the average seasonal values for each of the $L$ seasons. The $L$ averages are found by computing the average of the detrended values for the corresponding season. For example, for quarter 1,

\[
\bar{S}_{[1]} = \frac{S_1 + S_5 + S_9 + S_{13}}{4} = \frac{(-11.8309) + (-14.7544) + (-15.6779) + (-14.6015)}{4} = -14.2162
\]
Procedures of Additive Holt-Winters Method

• **Step 2**: Find the initial seasonal factors

4. Compute the average of the $L$ seasonal factors. The average should be 0.
Procedures of Additive Holt-Winters Method

• **Step 3:** Calculate a point forecast of $y_1$ from time 0 using the initial values

\[
\hat{y}_{T+p}(T) = \ell_T + pb_T + sn_{T+p-L} \quad (T = 0, p = 1)
\]

\[
\hat{y}_1(0) = \ell_0 + b_0 + sn_{1-4} = \ell_0 + b_0 + sn_{-3}
\]

\[
= 20.85 + 0.9809 + (-14.2162) = 7.6147
\]
Procedures of Additive Holt-Winters Method

• **Step 4:** Update the estimates $\ell_T$, $b_T$, and $sn_T$ by using some predetermined values of smoothing constants.

• Example: let $\alpha = 0.2$, $\gamma = 0.1$, and $\delta = 0.1$

\[
\ell_1 = \alpha(y_1 - sn_{1-4}) + (1 - \alpha)(\ell_0 + b_0)
\]
\[
= 0.2(10 - (-14.2162)) + 0.8(20.85 + 0.9808) = 22.3079
\]

\[
b_1 = \gamma(\ell_1 - \ell_0) + (1 - \gamma)b_0
\]
\[
= 0.1(22.3079 - 20.85) + 0.9(0.9809) = 1.0286
\]

\[
sn_1 = \delta(y_1 - \ell_1) + (1 - \delta)sn_{1-4}
\]
\[
= 0.1(10 - 22.3079) + 0.9(-14.2162) = -14.0254
\]

\[
\hat{y}_2(1) = \ell_1 + b_1 + sn_{2-4} = \ell_1 + b_1 + sn_{-2}
\]
\[
= 22.3079 + 1.0286 + 6.5529 = 29.8895
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Forecast Growth: 5424.06
Forecast Seasonal Factor: 0.9296
Forecast Made Last Period: 6.5240
Forecast Error: 40.3985
Forecast Error Squared: 1.5778

Slide 69
Procedures of Additive Holt-Winters Method

- **Step 5**: Find the most suitable combination of $\alpha$, $\gamma$, and $\delta$ that minimizes SSE (or MSE)
- **Example**: Use Solver in Excel as an illustration
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Additive Holt-Winters Method

• $p$-step-ahead forecast made at time $T$

$$\hat{y}_{T+p}(T) = \ell_T + pb_T + sn_{T+p-L} \quad (p = 1, 2, 3,...)$$

• Example

$$\hat{y}_{17}(16) = \ell_{16} + b_{16} + sn_{17-4} = 36.3426 + 0.9809 - 14.2162 = 23.1073$$

$$\hat{y}_{18}(16) = \ell_{16} + 2b_{16} + sn_{18-4} = 36.3426 + 2(0.9809) + 6.5529 = 44.8573$$

$$\hat{y}_{19}(16) = \ell_{16} + 3b_{16} + sn_{19-4} = 36.3426 + 3(0.9809) + 18.5721 = 57.8573$$

$$\hat{y}_{20}(16) = \ell_{16} + 4b_{16} + sn_{20-4} = 36.3426 + 4(0.9809) - 10.9088 = 29.3573$$
Additive Holt-Winters Method

- Example

Forecast Plot for Mountain Bike Sales

![Forecast Plot for Mountain Bike Sales](image)
Chapter Summary

• Simple Exponential Smoothing
  – No trend, no seasonal pattern

• Holt’s Trend Corrected Exponential Smoothing
  – Trend, no seasonal pattern

• Holt-Winters Methods
  – Both trend and seasonal pattern
    • Multiplicative Holt-Winters method
    • Additive Holt-Winters Method