Optimal Weight Choice for Frequentist Model Average Estimators

Hua Liang, Guohua Zou, Alan T. K. Wan, and Xinyu Zhang

There has been increasing interest recently in model averaging within the frequentist paradigm. The main benefit of model averaging over model selection is that it incorporates rather than ignores the uncertainty inherent in the model selection process. One of the most important, yet challenging, aspects of model averaging is how to optimally combine estimates from different models. In this work, we suggest a procedure of weight choice for frequentist model average estimators that exhibits optimality properties with respect to the estimator’s mean squared error (MSE). As a basis for demonstrating our idea, we consider averaging over a sequence of linear regression models. Building on this base, we develop a model weighing mechanism that involves minimizing the trace of an unbiased estimator of the model average estimator’s MSE. We further obtain results that reflect the finite sample as well as asymptotic optimality of the proposed mechanism. A Monte Carlo study based on simulated and real data evaluates and compares the finite sample properties of this mechanism with those of existing methods. The extension of the proposed weight selection scheme to general likelihood models is also considered. This article has supplementary material online.

KEY WORDS: Asymptotic optimality; Finite sample properties; Mallows criterion; Smoothed AIC; Smoothed BIC; Unbiased MSE estimate.

1. INTRODUCTION

It has been recognized that model selection neglects the uncertainty associated with the selection process, hence inference based on the final model can be seriously misleading (Hjort and Claeskens 2003). Traditional model selection procedures pick the best model that can explain the data at hand according to some model assessment criteria. The investigator then proceeds as if this model has been decided upon a priori. Conditional on the model chosen, statistical inference is typically conducted based on the corresponding conditional distribution of the parameter estimators. Standard errors conventionally estimated under such circumstances are well known to underreport variability (Hjort and Claeskens 2003; Danilov and Magnus 2004b). Model averaging, on the other hand, provides a coherent mechanism for accounting for this model uncertainty through combining parameter estimates across different models. When working with a distribution that is unconditional on the selected model, it incorporates rather than ignores the uncertainty inherent in the model selection process.

Model averaging has long been a popular technique among Bayesian statisticians. Reviews of the relevant Bayesian literature can be found in the works of Hoeting et al. (1999), Raftery, Madigan, and Hoeting (1997). One major criticism of Bayesian model averaging is that the procedure typically involves mixing a large number of priors regarding the unknowns, and it is unclear what the consequences will be when some of the priors are in conflict. Despite a growing frequentist model averaging literature, it would be fair to say that frequentists remain a distinct minority among those who advocate model averaging. However, this imbalance may soon be reconciled with some significant progress made in the frequentist literature in recent years. Buckland, Burnham, and Augustin (1997), for example, proposed a frequentist model weighting method according to values of a model selection criterion; Yang (2001, 2003) developed an adaptive model averaging method; Yuan and Yang (2005) further built on this method by proposing a model screening step prior to implementing adaptive regression by mixing; Hjort and Claeskens (2003) established a local misspecification framework for studying properties of post-selection and model average estimators; and Leung and Barron (2006) discussed a mixture least squares estimator with weights depending on the risk characteristics of the mixture estimator. The recent monograph of Claeskens and Hjort (2008) provided a useful summary of some of the progress that has been made in this area.

In a recent article, Hansen (2007) proposed a frequentist model average estimator with weights obtained by a mini-

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average estimator. Now, let \( y \) be an \( n \times 1 \) vector, \( H \) be an \( n \times P \) matrix of full column rank, \( H_p \) be an \( n \times p \) (\( p \leq P \)) matrix comprising the first \( p \) columns of \( H \), and \( (\omega_1, \omega_2, \ldots, \omega_P)' \) be a weight vector. The model average estimator considered by Hansen is of the form

\[
\hat{\Theta}_{\text{mna}} = \sum_{p=1}^{P} \omega_p \left( (H_p'H_p)^{-1}H_p'y \right).
\]

(1)

It is readily seen that \( \hat{\Theta}_{\text{mna}} \) is the weighted sum of least squares estimators from a sequence of \( P \) strictly nested regression models, where the \( p \)th model uses the first \( p \) variables in \( H \) as regressors. One fundamental requirement on \( \hat{\Theta}_{\text{mna}} \), therefore, is that the regressors be ordered prior to estimation. Asymptotic results on the MMA estimator developed by Hansen (2007) rely crucially on this assumption, which poses a strong limitation to his approach.

This article proposes a new method of weight choice for frequentist model average (FMA) estimators. As a basis for demonstrating our idea and to facilitate comparisons with existing FMA estimators, we adopt the linear regression model as our main analytical framework, although as we shall see, extensions to a more general likelihood framework are also feasible. Our setup assumes that the underlying model contains a set of focus regressors, whose inclusion in the model is mandatory on theoretical or other grounds irrespective of statistical significance, and a set of auxiliary regressors whose inclusion is viewed as optional. The model containing the focus regressors only is referred to as the narrow model; the extended models are those that contain the focus regressors and possibly some or all of the auxiliary regressors. This is the same setup used in a number of recent articles on pretesting (e.g., Magnus and Durbin 1999; Danilov and Magnus 2004a, 2004b), and it bears similarity to the local misspecification setup of Hjort and Claeskens (2003), which also distinguishes between narrow and extended models. The mandatory inclusion of focus regressors does not lead to any loss of generality since the focus regressors can in practice contain an intercept term only. Our approach to model weight selection is based on the mean squared error (MSE) properties of the combined estimator. Specifically, we derive an exact unbiased estimator of the MSE of the model average estimator, and propose selecting the model weights that minimize the trace of the MSE estimate. Unlike that of Hansen (2007), our approach does not require the regressors to be ordered, and in contrast to most previously proposed weight selection schemes, our criterion is based on analytical finite sample justifications. Our approach is similar in spirit to that advocated by Leung and Barron (2006), except that they focused on the risk bound of the combined estimator, whereas we provide an explicit weight choice criterion together with an analysis of the asymptotic and finite sample properties of the FMA estimator that results from the proposed weight choice method. Weighting schemes based on smoothed AIC (S-AIC) and smoothed BIC (S-BIC) (Buckland, Burnham, and Augustin 1997) are special cases of our proposed method. Our simulation results show that the estimator arising from the proposed weight selection method, which we label OPT estimator, frequently achieves smaller risk in terms of squared error loss than Hansen’s (2007) MMA estimator and S-BIC weights. While the bulk of our analysis focuses on the linear regression model, we also show that a similar weight choice mechanism may be crafted under a more general likelihood framework.

The presentation of this article goes as follows. Section 2 describes the model and estimators. In Section 3, we derive unbiased estimators of the finite sample MSEs of the FMA estimators, along with an investigation of the finite sample and asymptotic properties of the proposed criterion. Section 4 reports results of a Monte Carlo study based on simulated as well as real data. Section 5 discusses the generalization of our method to general parametric models, and our conclusions are presented in Section 6. Proofs of results are contained in the Appendix.

2. MODEL SETUP AND ESTIMATORS

Consider the linear regression model

\[
y = X\beta + Z\gamma + \epsilon, \quad \epsilon \sim N(0, \sigma^2I_n),
\]

(2)

where \( y \) is a vector of observations, \( X \) is a nonrandom regressor matrices, and \( \beta \) is a parameter vector. Additionally, we assume that \( H \equiv (X:Z) \) has full column rank \( k + m \). Here, \( X \) contains the focus regressors that must be included in the model, while \( Z \) contains the auxiliary or doubtful regressors whose inclusion in the model is optional. There is no loss of generality with this setup because \( X \) can contain no regressor other than an intercept term, or even be an empty matrix.

With \( m \) auxiliary regressors in \( Z \), there are a maximum of \( 2^m \) extended models to choose between. Let \( N \) be the number of extended models embodied in the model selection/averaging process. If all extended models are considered, then \( N = 2^m \). The case where only the full and narrow models are relevant corresponds to \( N = 2 \), and if we consider the nested setup as in the work of Hansen (2007), then \( N = m + 1 \). Traditional model selection procedures pick the best model that can explain the data at hand based on some model assessment criteria, while model averaging combines estimates obtained from different models. Although one can argue that model selection may be more attractive than model averaging when the true model is a candidate model, this should not be used as a criticism against the basic principle underlying model averaging. The reason is that there is always the possibility of selecting the wrong or even a very poor model. With model averaging, it is not necessary to assume that the true model is among the set of considered models, while such is not the case in typical model selection studies.

Under the above setup, the fully restricted (i.e., \( \gamma = 0 \)) and unrestricted least squares estimators of \( \beta \) are \( \hat{\beta}_r = (X'X)^{-1}X'y \) and \( \hat{\beta}_u = \hat{\beta}_r - (X'X)^{-1}X'Z(Z'MZ)^{-1}Z'My \), respectively, where \( M = I_n - X(X'X)^{-1}X' \), and the unrestricted least squares estimator of \( \gamma \) is \( \hat{\gamma} = (Z'MZ)^{-1}Z'My \). Now, let \( \theta = (Z'MZ)^{1/2} \gamma \) and \( \hat{\theta} = (Z'MZ)^{1/2}\hat{\gamma} \). Note that \( \hat{\theta} \sim N(\theta, \sigma^2I_m) \); in particular, \( \hat{\theta} \) has a covariance matrix that is a scalar multiple of an identity matrix. Hence, it is of analytical convenience to write \( \hat{\beta}_u = \hat{\beta}_r - Q\hat{\theta} \), where \( Q = (X'X)^{-1}X'Z(Z'MZ)^{-1} \). In conformity, the ith \( (1 \leq i \leq 2^m) \) partially restricted least squares estimator of \( \beta \) can be written as \( \hat{\beta}_{i} = \hat{\beta}_r - QW_i\hat{\theta} \), where \( W_i = I_m - P_i \), \( P_i = (Z'MZ)^{-1/2}S_i(Z'MZ)^{-1}S_i' \), and \( S_i \) is
an $m \times m$ symmetric idempotent matrix of rank $r_i \geq 0$, and $S_j$ is an $m \times r_1$ selection matrix of rank $r_1$ so that $S_j' = (I_{r_1}, 0)$ or a column permutation thereof (Danilov and 2004b). The $i$th partially restricted least squares estimator of $\gamma$ is $\hat{\gamma}_i = (Z'MZ)^{-1/2}W_i\hat{\beta}$ under the restriction $S_j'\gamma = 0$. Further, let $\sigma_i^2 = \|y - X\hat{\beta}_i - Z\hat{\gamma}_i\|^2/n$ be the estimator of $\sigma^2$ under the unrestricted model. Then, an FMA estimator of $\beta$ in model (2) may be written as
\[
\hat{\beta}_f = \sum_{i=1}^{N} \lambda_i \hat{\beta}_i.
\] (3)
with weights satisfying $\lambda_i \geq 0$ and $\sum_{i=1}^{N} \lambda_i = 1$.

Here, we concentrate on random weights. Specifically, we let $\lambda_i$ depend on $\hat{\theta}$ and $\hat{\sigma}^2$. This consideration is motivated by the following observation. Let $q_i$ be the number of regressors in the $i$th partially restricted model; then the AIC of the $i$th model is $AIC(i) = n \log(\hat{\sigma}_i^2) + 2q_i + 1$, where $\hat{\sigma}_i^2 = \|y - X\hat{\beta}_i - Z\hat{\gamma}_i\|^2/n$ is the maximum likelihood estimator of $\sigma^2$ under the $i$th model. Note that we can write $\hat{\sigma}_i^2 = (n - k + m)\hat{\sigma}^2 + n/\hat{\theta}P_1\hat{\theta}/n$. Putting the latter expression of $\hat{\sigma}_i^2$ in the AIC expression of the $i$th model, we observe that the AIC depends on the data only through $\hat{\theta}$ and $\hat{\sigma}^2$. Motivated by this, we consider weights $\lambda_i = \lambda_i(\hat{\theta}, \hat{\sigma}^2)$ that only depend on $\hat{\theta}$ and $\hat{\sigma}^2$. Writing $W = \sum_{i=1}^{N} \lambda_i W_i$, we can rewrite the FMA estimator as $\hat{\beta}_f = \hat{\beta}_i - QW\hat{\theta}$.

Note that the model considered by Hansen (2007) makes no distinction between focus and auxiliary regressors, and may be stated using our notations as $y = H\theta + u$, where $\theta = (\beta', \gamma')'$ is a vector of coefficients, $u = Bu + e$, $B$ is a set of omitted regressors, and $v$ is the corresponding coefficient vector. Hansen (2007) concentrated on the estimation of $\mu^* = H\theta + Bu$, and evaluated the performance of $\hat{\Omega}_{\text{mama}}$, the MMA estimator of $\Omega$, in terms of the criterion $R^* = E[|H\hat{\Omega}_{\text{mama}} - \mu^*|^2]$, which is the risk of the estimator of the prediction vector.

3. UNBIASED ESTIMATION OF MSE AND OPTIMAL WEIGHT CHOICE

In this section we consider the choice of weights in (3). Our weight choice method is based on an MSE minimizing estimation objective that is designed to provide MSE improvements over other FMA estimators, especially in finite samples. Here, the MSE of $\hat{\beta}_f$ is defined as the matrix $E(\hat{\beta}_f - \beta)(\hat{\beta}_f - \beta)'$, with its diagonal elements being the MSEs of the estimators for the individual components of $\beta$. The trace of the MSE matrix is equal to the expected squared error loss function $E\|\hat{\beta}_f - \beta\|^2$, known also as the risk of the estimator under squared error loss, or the weak MSE accuracy measure (Wallace 1972).

To make our concept operational, we first derive an unbiased estimator of the MSE of $\hat{\beta}_f$. The optimal model average weights are then obtained by minimizing the trace of the MSE estimate. With appropriate modifications we also generalize the method to the derivation of the MMA estimator of the predictor $\hat{\mu}_f = H\hat{\theta}_f$, where $\hat{\theta}_f = (\hat{\beta}_f', \hat{\gamma}_f')'$, and $\hat{\gamma}_f = \sum_{i=1}^{N} \lambda_i(\hat{\theta}, \hat{\sigma}^2)\hat{\gamma}_i(\hat{\theta}, \hat{\sigma}^2)$ is the FMA estimator of $\gamma$ corresponding to $\hat{\beta}_f$.

3.1 An Unbiased Estimator of the MSE of $\hat{\beta}_f$

Our principal result is stated in the following theorem:

Theorem 1. Under model (2), assuming that $\lambda_i(\hat{\theta}, \hat{\sigma}^2)$, $i = 1, \ldots, N$, are continuous functions with piecewise continuous partial derivatives with respect to $\hat{\theta}$, and provided that the expectations of $(\partial\lambda_i(\hat{\theta}, \hat{\sigma}^2)/\partial\hat{\theta})\hat{\theta}$ and $\Psi$ (defined below) exist, an unbiased estimator of the MSE of the FMA estimator $\hat{\beta}_f$ is given by
\[
\text{MSE}(\hat{\beta}_f) = \hat{\sigma}^2(X'X)^{-1} - \hat{\sigma}^2QQ' + \{Q(I_m - W)\hat{\theta}\hat{\theta}'\}
\]
\[
+ \Psi(\hat{\theta}, \hat{\sigma}^2) \times \int_0^{\hat{\sigma}^2} (n - k + m)/2 - 1\Psi_1(\hat{\theta}, t) dt,
\] (4)
where $A^{\otimes 2} = AA'$ for any vector or matrix $A$, and
\[
\Psi(\hat{\theta}, \hat{\sigma}^2) = (n - k - m)/2(\hat{\sigma}^2)^{-n-k-2m/2+1}
\]
\[
\times \left\{ (n - k - m)/2 - 1\Psi_1(\hat{\theta}, t) dt. \right\}
\] (5)

Proof. See the Appendix.

The unbiased estimator of the MSE of $\hat{\beta}_f$ provides a basis for measuring the estimator’s precision that can be justified in finite samples. The goal of our criterion is to choose $\lambda_i$’s in (3) that minimize the trace of $\text{MSE}(\hat{\beta}_f)$. Now, from (4), it is straightforward to show that the trace of $\text{MSE}(\hat{\beta}_f)$ is
\[
\text{tr}(\hat{\beta}_f (I_m - W)\hat{\theta}) + 2 \text{tr}((\hat{\theta} - \hat{\beta}_f)^2).
\] (6)
However, the practical application of (7) is limited to some degree by the complexity of the term $\Psi(\hat{\theta}, \hat{\sigma}^2)$, which is cumbersome to calculate. To get around this problem we replace $\Psi(\hat{\theta}, \hat{\sigma}^2)$ by an approximate quantity. Note that under the conditions of Theorem 1, $E_{\hat{\sigma}^2}(\Psi(\hat{\theta}, \hat{\sigma}^2)) = \sigma^2 E_{\hat{\sigma}^2}(\Psi_1(\hat{\theta}, \hat{\sigma}^2))$, where $E_{\hat{\sigma}^2}$ denotes expectation only with respect to $\hat{\sigma}^2$ [see (A.4) and its proof in the Appendix for details]. Thus, it appears reasonable to replace $\Psi(\hat{\theta}, \hat{\sigma}^2)$ by $\hat{\sigma}^2 \Psi_1(\hat{\theta}, \hat{\sigma}^2)$ in (7). This results in the following approximate risk quantity:
\[
\text{tr}(\hat{\beta}_f (I_m - W)\hat{\theta}) + 2 \text{tr}((\hat{\theta} - \hat{\beta}_f)^2).
\] (8)
The online supplementary material provides results which show that the values produced by $\hat{\sigma}^2 \Psi_1(\hat{\theta}, \hat{\sigma}^2)$ typically accord with those of $\Psi(\hat{\theta}, \hat{\sigma}^2)$ very closely.

We note that if the FMA estimator $\hat{\beta}_f$ is based on the S-AIC weights, then $\lambda_i = e^{-\Psi(\hat{\sigma}_i^2)}/\sum_{j=1}^{N} e^{-\Psi(\hat{\sigma}_j^2)-n/2}$. If the S-BIC weights are used, then $\lambda_i = n^{-n/2}(\hat{\sigma}_i^2)^{-n/2}/\sum_{j=1}^{N} n^{-n/2} \times (\hat{\sigma}_j^2)^{-n/2}$. Also, if one uses the smoothed residual mean squares (S-RMS) weights (Bates and Granger 1969), then $\lambda_i = (n - q_i)(\hat{\sigma}_i^2)^{-1}/\sum_{j=1}^{N} (n - q_j)(\hat{\sigma}_j^2)^{-1}$. A natural generalization of these weights is found in the following class of random weights:
\[
\lambda_i(\hat{\theta}, \hat{\sigma}^2) = \frac{a^n(n - q_i)^b(\hat{\sigma}_i^2)^c}{\sum_{j=1}^{N} a^n(n - q_j)^b(\hat{\sigma}_j^2)^c},
\] (9)
where \( a (> 0) \), \( b (\geq 0) \), and \( c (\leq 0) \) are constants. The S-AIC weights are obtained by setting \( a = e^{-1} \), \( b = 0 \), and \( c = -n/2 \); the S-BIC weights result when \( a = n^{-1/2} \), \( b = 0 \), and \( c = -n/2 \); and when \( a = b = 1 \) and \( c = -1 \), \( \lambda_j(\hat{\theta}, \hat{\sigma}^2) \) reduces to the S-RMS weights. Further, by setting \( a = 1 \), \( b = 2 \), and \( c = -1 \), we obtain weights that correspond to the smoothed generalized cross-validation of Craven and Wahba (1979), which is almost identical to the average prediction MSE criterion due apparently to J. M. Tukey (see Wetherill et al. 1986, p. 243; and Leeb and Pötscher 2008, p. 898). With slight modifications, weights corresponding to the smoothed version of Hurvich and Tsai’s (1989) bias-corrected AIC can also be written as a special case of (9).

Now, define \( L = (l_{ij}) \) and \( G = (g_{ij}) \), where \( l_{ij} = \hat{\theta}'(I_m - W_i)'Q(I_m - W_j)\hat{\theta} \) and \( g_{ij} = (\hat{\sigma}^2)^{-1}\hat{\theta}'W_i'Q(I_m - W_j)\hat{\theta} \) for \( i, j = 1, \ldots, N \). Additionally, let \( g \) and \( \phi \) each be an \( N \times 1 \) vector with \( g \) consisting of the diagonal elements of \( G \) and the \( i \)-th element of \( \phi \) being \( tr(QW_iQ) \), \( i = 1, \ldots, N \). Recognizing that

\[
\partial \lambda_j(\hat{\theta}, \hat{\sigma}^2)/\partial \hat{\sigma} = (2/n)c\lambda_j(\hat{\theta}, \hat{\sigma}^2) \left\{ (\hat{\sigma}^2)^{-1}(I_m - W_i) \right\}^{(j-1)}(I_m - W_j)
\]

putting (10) in \( \Psi_1(\hat{\theta}, \hat{\sigma}^2) \) given by (6) and using (8), we have

\[
\tilde{R}_{\hat{\theta}}(\tilde{\sigma}^2) = \hat{\sigma}^2 tr(X'X)^{-1} - \hat{\sigma}^2 tr(\psi_1) + \lambda'(a, b, c)\mu_\lambda(a, b, c)
\]

\[
- (4/n)c\hat{\sigma}^2\lambda'(a, b, c)G\lambda(a, b, c) + 2\hat{\sigma}^2\lambda'(a, b, c)\phi
\]

\[
+ (4/n)c\hat{\sigma}^2\lambda'(a, b, c)g,
\]

where \( \lambda(a, b, c) \) is an \( N \times 1 \) vector comprising \( \lambda_i(\hat{\theta}, \hat{\sigma}^2) \), \( i = 1, \ldots, N \). We use \( a, b, \) and \( c \), defined in Equation (9), as arguments of \( \lambda \) to emphasize the role of these parameters in the optimal weight choice. The primary goal is to select the appropriate values of \( a, b, \) and \( c \) in \( \lambda(a, b, c) \) that minimize (11). Let \( \lambda(a^*, b^*, c^*) \) be such a vector. We call the estimator \( \tilde{\beta}_{\hat{\theta}} \) corresponding to \( \lambda(a^*, b^*, c^*) \) the optimal FMA estimator (labeled as OPT estimator hereafter).

Before proceeding further we want to draw readers’ attention to the following special case. Suppose we set \( c = 0 \) in (11); then the weights \( \lambda_i's \) in (9) reduce to deterministic weights. For this special case, if we consider mixing only \( \tilde{\beta}_{\hat{\theta}} \) and \( \tilde{\beta}_{\hat{\sigma}} \) (i.e., all the regressors in \( Z \) are either in or out), then minimizing (11) with respect to \( a, b \) leads to the estimator

\[
\tilde{\beta}_{\mu} = \left\{ 1 - \frac{\hat{\sigma}^2 tr(\psi_1)}{||\tilde{\beta}_{\hat{\theta}} - \tilde{\beta}_{\hat{\sigma}}||^2} \right\} \tilde{\beta}_{\hat{\theta}} + \frac{\hat{\sigma}^2 tr(\psi_1)}{||\tilde{\beta}_{\hat{\theta}} - \tilde{\beta}_{\hat{\sigma}}||^2} \tilde{\beta}_{\hat{\sigma}},
\]

which turns out to be the James–Stein type estimator studied by Kim and White (2001). In view of the fact that it is obtained from minimizing (11), \( \tilde{\beta}_{\mu} \) is also an optimal estimator by our criterion if one restricts attention to the subspace of \( c = 0 \). Clearly, the OPT estimator that minimizes (11) (regardless of the value of \( c \)) has optimal properties among a broader class of estimators. In this sense, \( \tilde{\beta}_{\mu} \) is suboptimal with respect to our criterion when compared to the OPT estimator.

### 3.2 An Unbiased Estimator of the MSE of \( \hat{\mu}_f \)

With some efforts the preceding framework of finding optimal weights may be generalized to encompass the estimation of the vector \( \mu = H\theta \), which is the conditional mean prediction of the dependent variable. Denote the FMA estimator of \( \mu \) as

\[
\hat{\mu}_f = H\hat{\theta}_f = H \sum_{i=1}^{N} \lambda_i(\hat{\theta}, \hat{\sigma}^2)\hat{\theta}(i) \equiv \sum_{i=1}^{N} \lambda_i(\hat{\theta}, \hat{\sigma}^2)\hat{\mu}(i),
\]

where \( \hat{\mu}(i) = H\hat{\theta}(i) \), and \( \hat{\theta}(i) \) is the \( i \)-th partially restricted least squares estimator of \( \theta \). Note that when there is no mandatory regressor, that is, \( k = 0 \), \( \beta \) does not exist, and thus there is no FMA estimator of \( \beta \); on the other hand, the average of \( \hat{\mu}(i) \) exists whether or not there are focus regressors in the model. In this case, \( \hat{\theta}(i) \) reduces to \( \hat{\gamma}(i) \), and \( \hat{\theta} = (Z'Z)^{-1/2}Z'y \).

**Theorem 2.** Under model (2) and the same conditions as in Theorem 1, an unbiased estimator of the MSE of \( \hat{\mu}_f \) is given by

\[
\text{MSE}(\hat{\mu}_f) = \hat{\sigma}^2 X'(X'X)^{-1}X' + \phi(\hat{\sigma}, \hat{\sigma}^2, XQ, (XQ)') - \phi(\hat{\sigma}, \hat{\sigma}^2, Z(Z'MZ)^{-1/2}, (XQ)') - \phi(\hat{\sigma}, \hat{\sigma}^2, Z(Z'MZ)^{-1/2}, (Z(Z'MZ)^{-1/2})),
\]

where

\[
\phi(\hat{\sigma}, \hat{\sigma}^2, D_1, D_2) = -\hat{\sigma}^2 D_1 D_2 + D_1 (I_m - W)\hat{\theta}(i)^{\otimes 2} D_2
\]

\[
+ D_1 \Xi(\hat{\theta}, \hat{\sigma}^2) D_2
\]

\[
+ D_1 \Xi(\hat{\theta}, \hat{\sigma}^2) D_2,
\]

\[
\Xi(\hat{\theta}, \hat{\sigma}^2) = (n - k - m)/2(\hat{\sigma}^2)^{-(n - k - m)/2 + 1}
\]

\[
\times \int_0^{\hat{\sigma}^2} (n - k - m)/2 - 1 \Xi(\hat{\theta}, t) dt,
\]

and

\[
\Xi(\hat{\theta}, t) = W + \sum_{i=1}^{N} \partial \lambda_j(\hat{\theta}, t)/\partial \hat{\theta} \hat{\theta}'W_i.
\]

**Proof.** See the Appendix.

The trace of \( \text{MSE}(\hat{\mu}_f) \) is

\[
\text{tr}(\text{MSE}(\hat{\mu}_f)) = k\hat{\sigma}^2 + \text{tr}(\phi(\hat{\sigma}, \hat{\sigma}^2, XQ, (XQ)')) - 2 \text{tr}(\phi(\hat{\sigma}, \hat{\sigma}^2, Z(Z'MZ)^{-1/2}, (XQ)')) + \text{tr}(\phi(\hat{\sigma}, \hat{\sigma}^2, Z(Z'MZ)^{-1/2}, [Z(Z'MZ)^{-1/2}]'),
\]

Note that all terms except the last in (14) and (18) vanish when no regressor is considered mandatory.

To overcome the computational difficulties associated with (18), and noting that \( E_{\hat{\sigma}^2}(\Xi(\hat{\theta}, \hat{\sigma}^2)) = \hat{\sigma}^2 E_{\hat{\sigma}^2}(\Xi(\hat{\theta}, \hat{\sigma}^2)) \) under the conditions of Theorem 1, we replace \( \Xi(\hat{\theta}, \hat{\sigma}^2) \) by \( \hat{\sigma}^2 \Xi(\hat{\theta}, \hat{\sigma}^2) \) in (15). This is analogous to the substitution of \( \Psi(\hat{\theta}, \hat{\sigma}^2) \) by \( \hat{\sigma}^2 \Psi(\hat{\theta}, \hat{\sigma}^2) \) in (8). Following this substitution we
obtain the following approximately unbiased estimator of the trace of the MSE of \( \hat{\mu}_f \):

\[
\hat{R}_d(\hat{\mu}_f) = k\hat{\sigma}^2 + tr\{\hat{\psi}_1(\hat{\theta}, \hat{\sigma}^2, XQ, (XQ')\}
- 2tr\{\hat{\psi}_1(\hat{\theta}, \hat{\sigma}^2, Z(Z'MZ)^{-1/2}, (XQ'))\}
+ tr\{\hat{\psi}_1(\hat{\theta}, \hat{\sigma}^2, Z(Z'MZ)^{-1/2}, (Z(Z'MZ)^{-1/2}))\},
\]

(19)

where \( \hat{\psi}_1(\hat{\theta}, \hat{\sigma}^2, D_1, D_2) \) has the same expression as \( \psi(\hat{\theta}, \hat{\sigma}^2, D_1, D_2) \), except that \( \mathbb{E}(\hat{\theta}, \hat{\sigma}^2) \) in \( \psi(\cdot) \) is replaced everywhere by \( \hat{\sigma}^2\mathbb{E}(\hat{\theta}, \hat{\sigma}^2) \).

Again, let the weight \( \lambda_i(\hat{\theta}, \hat{\sigma}^2) \) be of the form (9). Note that \( P_j \) is symmetric idempotent and \( \{\hat{\theta}(\cdot) - \hat{\theta}\} \) with \( \hat{\theta} = (\hat{\theta}_j) \) as the value of \( (a, b, c) \) that minimizes \( \hat{R}_d(\hat{\mu}_f(\lambda(a, b, c))) \) with \( (a, b, c) \in \mathcal{D} \). Assume also that \( \lambda(\hat{\theta}, \hat{\sigma}^2) \) belongs to \( \mathcal{D}_0 \).

**Theorem 3.** When \( n \to \infty \), provided that the conditions

\[
\mu'\mu = O(n)
\]

and

\[
\xi_n^{-2} \xi_n \to 0
\]

are satisfied, then

\[
\frac{L_n(\lambda(a, b, c))}{\inf_{(a, b, c) \in \mathcal{D}_0} L_n(\lambda(a, b, c))} \xrightarrow{p} 1.
\]

**Proof.** See the Appendix.

Theorem 3 states that subject to the fulfillment of conditions (21) and (22), the large sample squared error associated with the OPE estimator converges in probability to the smallest achievable squared error of any FMA estimator in the form of (13) based on model weights given in (9), with values of \( (a, b, c) \) restricted to the subset \( \mathcal{D}_0 \).

The following discusses the relevance of the subset \( \mathcal{D}_0 \) and conditions (21) and (22). First, note that condition (21) is a common condition concerning the sum of \( \mu_j^2, j = 1, \ldots, n \) (e.g., Shao 1997). Under (21), we have

\[
\hat{\sigma}_i^2 = (y'My - \hat{\theta}'W\hat{\theta})/n \leq y'y/n
= (\mu'\mu + \varepsilon'\varepsilon + 2\mu'\varepsilon)n = O_p(1)
\]

for any submodel \( i \). Discounting the cases of \( \mathcal{U} \) being an empty set and \( \mathcal{U} = \{1, \ldots, N\} \), by the result that for any \( \tau \in \mathcal{U}, \hat{\sigma}_i^2 \xrightarrow{p} \sigma^2 > 0 \), we have, for any submodel \( \tau \in \mathcal{U} \) and \( i \notin \mathcal{U} \),

\[
\hat{\sigma}_i^2 / \hat{\sigma}_i^2 = O_p(1).
\]

Combining (9), (25) and the restriction of \( c \leq 0 \), it can be seen that for any submodels \( \tau \in \mathcal{U}, i \notin \mathcal{U} \) and any \( (a, b, c) \in \mathcal{D} \), we have \( \lambda_\tau(a, b, c)/\lambda_\tau(a, b, c) = O_p(1) \), and thus

\[
\sum_{\tau \in \mathcal{U}} \lambda_\tau(a, b, c) / \left(1 - \sum_{\tau \in \mathcal{U}} \lambda_\tau(a, b, c)\right)
\leq \sum_{\tau \in \mathcal{U}} \lambda_\tau(a, b, c)/\lambda_\tau(a, b, c)
= O_p(1).
\]
Consequently, \( \sum_{\tau \in D} \lambda_{\tau}(a, b, c) \) cannot tend to 1 in probability. This means that in large samples, the model weighting scheme stipulated by (9) cannot give rise to zero weight for all biased models. In other words, this weighting scheme implies that even with large samples, at least one of the models that form the FMA estimator must be a biased model. This is why the subset \( D_0 \) for which \( \sum_{\tau \in D} \lambda_{\tau}(a, b, c) \leq 1 - \rho \) is relevant. Obviously, when \( \rho \) is very small, \( D_0 \) is very close to \( D \).

As an aside, it is of interest to mention that if there exist constants \( k_1 \) and \( k_2 \) such that \( 0 < k_1 \leq k_2 < \infty \), and \( k_1 \leq \bar{\sigma}_i^2 / \sigma^2 \leq k_2 \) with probability 1 for any \( i \in \{1, \ldots, N\} \) (see, e.g., Yang 2003 and Yuan and Yang 2005 for similar conditions), and constants \( \bar{\alpha}_1 \), \( \bar{\alpha}_2 \), and \( \bar{b} \) such that \( 0 < \bar{\alpha}_1 \leq \bar{\alpha}_2 < \infty \), \( 0 \leq b \leq \bar{b} < \infty \), then \( \sum_{\tau \in D} \lambda_{\tau}(a, b, c) \leq 1 - \rho \) is always true with probability 1, provided that \( U \) is not the set \( \{1, \ldots, N\} \). The proof is available from the online supplementary material. It is instructive to note that \( \sum_{\tau \in D} \lambda_{\tau}(a, b, c) \leq 1 - \rho \) automatically holds for the case of \( U \) being empty.

The validity of Theorem 3 also depends on condition (22). A necessary condition for (22) to hold is

\[
\xi_n \to \infty. \tag{26}
\]

Condition (26) is in fact very mild, and thus likely to be fulfilled in practice because \( W \) does not include any weight vector that assigns nonzero weights only to the unbiased models. Now, if \( \xi_n \to \infty \) holds, then \( \xi_n^{-2} \xi_n \to 0 \) also holds as long as \( \xi_n \) tends to infinity at a rate slower than that of \( \xi_k \) to infinity. One can expect the rate of \( \xi_n \to \infty \) to reduce if some of the very poor models that are associated with large risks are removed at the outset. Thus, it seems desirable to combine over an optimal subset rather than the full set of models. The model screening step developed by Yuan and Yang (2005) may be useful in this regard.

The following simple example sheds further light on condition (22). Consider Equation (2) with \( \beta = 1, y = 0.1, X = 4_n \) being an \( n \times 1 \) vector of ones, and \( Z = (\cos(\frac{2\pi}{n}), \cos(\frac{4\pi}{n}), \ldots, \cos(\frac{2n\pi}{n})) \). Under this setup, there is only one restricted model in addition to the unrestricted model as candidates for model combination. We show in the online supplementary material that when \( n \geq 2000\sigma^2 \), \( \xi_n = 0.005n + \sigma^2 \) and \( \xi_n \geq \sigma^2 \xi_n \). Condition (22) (22) therefore holds. The online supplementary material provides more theoretical examples concerning the relevance of condition (22).

An attempt is also made to verify condition (22) by simulations based on a model setup similar to that of Example 1 in Section 4. We consider the case of three auxiliary regressors, and let \( \Theta = (5, 8, 7, c_3(1, 0, 1))^t \). The importance of the auxiliary regressors relative to the focus regressors is measured by the ratio \( \alpha = \text{var}(\sum_{j=k}^n \theta_j x_{ji}) / \text{var}(\sum_{j=k}^3 \theta_j x_{ji}) \). We set \( c_3 \) to values that correspond to \( \alpha = 0.1, 0.5 \), and 0.9, and \( \rho \) in the weight set \( W \) to 0.3. The simulation results as shown in Figure 1 indicate that \( \xi_n^{-2} \xi_n \) converges to 0 from above as the sample size increases, and an increase in \( \alpha \) has the effect of speeding up the convergence of \( \xi_n^{-2} \xi_n \) to zero, ceteris paribus.

It is instructive to mention that the restriction of \( \rho > 0 \) is needed only for condition (22) to hold; once (21) and (22) are established, our subsequent steps for proving (23) do not explicitly require \( \rho > 0 \). We suspect that (23) also holds without having to invoke the assumption of \( \rho > 0 \) as an underlying condition. However, if the nonzero restriction on \( \rho \) is relaxed, the technical challenge for establishing (23) will be formidable since one then has to bypass condition (22). The development of such a proof technique is left for future research.

4. SIMULATION STUDIES

In this section, we conduct simulation experiments to compare the finite sample performance of the OPT estimator with the MMA estimator and the model average estimators based on the S-AIC and S-BIC weights (hereafter referred to as the S-AIC and S-BIC estimators, respectively). Example 1 is based on Equation (2), with the MMA estimator obtained using a regressor ordering pattern decided a priori, and all other FMA estimators combined across \( 2^n \) submodels. Example 2, which is based on the same setting as one of the experiments in the article by Hansen (2007), examines the performance of the OPT estimator when it combines models in the same manner as the MMA estimator. The purpose is to evaluate the OPT estimator relative to the MMA estimator when both are considered on a platform that is supposed to favor the MMA estimator. In Example 3, we further demonstrate the advantages of the OPT estimator over the MMA estimator using a real dataset taken from the work of Danilov and Magnus (2004a). The main objective is to examine the extent to which different patterns of regressor ordering affect the properties of the MMA estimator. The results of Example 3 show that the performance of the MMA estimator is highly sensitive to the pattern of regressor ordering, thus illustrating the viability of the OPT estimator as an alternative.

It also appears important to emphasize that the implementation of the OPT method is based on the set \( \mathcal{D} \), not \( \mathcal{D}_0 \). The latter subset is only relevant for the optimality theory, not for empirical application of the OPT method. In fact, \( \mathcal{D}_0 \) cannot be determined in practice because one cannot know which of the candidate models are unbiased when the true model is unknown. However, as mentioned before, this does not render the
theoretical results of the previous section irrelevant because $\rho$
can take on any positive value close to zero; hence $D_0$ is
generally very close to $D$. In all cases of our simulations, we set
the value of $\hat{c}$ to $n/2$ so that $D$ can encompass the S-AIC
and S-BIC weights.

Example 1. The data are generated from the model

$$y_i = \sum_{j=1}^{10} \theta_j x_{ji} + e_i,$$

where $x_{1j} = 1$, $x_{ji} \sim N(0, 1)$ for $j = 2, \ldots, 10$, and $e_i \sim N(0, 1)$,
$i = 1, \ldots, n$. The sample size varies between $n = 30$, 80, 150,
and 300. The error term $e_i$ is independent of $x_{ji}$'s, and all
$x_{ji}$'s are independent of one another. Arbitrarily, we let $x_{i1}, x_{i2},$
and $x_{i3}$ be the focus regressors, and consider all other regres-
sors as auxiliary. The parameters are given by $\theta = (\theta_1, \ldots,$
$\theta_{10})' = (1, c_1(3, 4, c_2(0.5, 0.6, 0, 1, 0.4, 0.3, 0.8))'$. Let $\alpha =
\text{var}(\sum_{j=1}^{10} \theta_j x_{ji}) / \text{var}(\sum_{j=1}^{10} \theta_j x_{ji})$, which may be written as $\alpha =
c^2/2(0.5^2 + 0.6^2 + 0^2 + 1^2 + 0.4^2 + 0.3^2 + 0.8^2)/(c_1^2(3^2 + 4^2)) =
c^2/10$ in the present context. Note that $\alpha$ measures the impor-
tance of the auxiliary regressors relative to the focus regressors;
the larger the value of $c^2$ (and hence $\alpha$), the greater the impor-
tance of the auxiliary regressors. We set $\alpha$ to 0.1 and 0.9. The
population $R^2 = 25c^2(1 + \alpha)/(1 + 25c^2(1 + \alpha))$ is controlled
by the parameter $c_1$, where $25c^2(1 + \alpha) = \text{var}(\sum_{j=1}^{10} \theta_j x_{ji})$ is the
variance of the linear combination of all regressors, focus and
auxiliary. We set $R^2$ in the range of [0.1, 0.9]. With seven auxiliary
regressors, the OPT, S-AIC, and S-BIC estimators average estimates
across $2^7$ models. In computing the MMA estimator, we order the regressors as $x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}, x_{i6}, x_{i7}, x_{i8}, x_{i9},$
and $x_{i10}$. The MMA estimator is then obtained by applying
Equation (1) with the model weights derived by minimizing the
Mallows criterion [see Hansen 2007, equation (11)]. Our simu-
lation experiment is based on 2000 replications.

We begin by discussing the results when the estimators are
evaluated in terms of risk under the loss function

$$L^{(1)} = \left| \hat{\mu} - \sum_{j=1}^{10} \theta_j x_{ji} \right|^2,$$

that is, the predictive loss of $\hat{\mu}$, where $x_j = (x_{j1}, \ldots, x_{jn})'$. With
the predictive loss as the penalty function, we obtain the OPT
estimator by selecting the $\lambda$ that minimizes (20). Results of risk
comparisons are given in Figures 2 and 3. As in the article by
Hansen (2007), we normalize the risk by dividing by the risk
of the infeasible optimal least squares estimator, that is, the risk
of the best-fitting model among the $2^7$ models. Figures 2 and 3
reveal that the OPT estimator generally has better risk perfor-
mance than the other three estimators no matter the value of $n$
and $\alpha$. Exceptions occur when $R^2$ is very large or small. For ex-
ample, when $R^2$ is near 0.1 and $n$ is small, the MMA estimator
typically achieves the lowest risk; when $R^2$ is near 0.9 and $n$
is large, the S-AIC and S-BIC estimators can be superior to both
the MMA and OPT estimators.

Next we consider the efficiency of the estimators of the coef-
ficients of the focus regressors. Evaluation is based on the loss
function

$$L^{(2)} = \sum_{j=1}^{3} (\hat{\theta}_j - \theta)^2.$$

In this case we compute the weight vector of the OPT estima-
tor by minimizing (11). Because the MMA approach does not
distinguish between focus and auxiliary regressors, it makes no
sense to include the MMA estimator in the evaluation, and thus

![Figure 2](image-url)
we compare only the OPT, S-AIC, and S-BIC estimators when interest focuses on the estimation of the coefficients of the focus regressors. Figure 4 provides a selection of results. Again, in each case the risk is normalized by dividing by the risk of the infeasible optimal least squares estimator. From Figure 4, we observe that when \( n \) and \( \alpha \) are both small, the S-BIC estimator has the best performance while the OPT estimator has the worst; however, when \( n \) is large or \( \alpha \) is large, except when \( R^2 \) is very large or very small, the OPT estimator is the best while the S-BIC is the worst. Results of cases not shown here are available in the online supplementary material; in general, they depict very similar characteristics to those shown in Figure 4.

**Example 2.** This example is based on the same setting as in the article by Hansen (2007), that is, \( y_i = \sum_{j=1}^{\infty} \theta_j x_{ji} + e_i \), \( x_{1i} = 1 \), all remaining \( x_{ji} \)'s are \( N(0, 1) \), \( e_i \) is distributed as

![Figure 3. Results for Example 1: risk comparisons under \( L^{(1)} \) loss when \( \alpha = 0.9 \).](image1)

![Figure 4. Results for Example 1: risk comparisons under \( L^{(2)} \) loss.](image2)
N(0, 1), independent of \( x_{ji} \)'s, all \( x_{ji} \)'s are independent of one another, \( \theta_j = c_3 \sqrt{2 \alpha_1}^{-\alpha_1 - 1/2} \), and the population \( R^2 = c_j^2/(1 + c_j^2) \) is controlled by \( c_j \). Sample size varies between \( n = 50, 150, 400, \) and \( 1000 \), \( \alpha_1 \) is set to 0.5, 1.0, and 1.5, and \( R^2 \) is set in the range of \([0.1, 0.9] \). The total number of regressors \( P \) in the regression is determined by \( P = 3n^{1/3} \). Like Hansen (2007), we consider \( P \) nested approximating submodels with the \( p \)th submodel comprising the first \( p \) regressors. All four model average estimators combine estimates across these \( P \) submodels. Note that although the OPT, S-AIC, and S-BIC can potentially combine estimates from all candidate models, we only consider \( P \) nested models here—the purpose is to evaluate the OPT estimator when all estimators are considered on a platform that is supposed to favor the MMA estimator. As in the article by Hansen (2007), evaluation is based on the loss function

\[
L^{(3)} = \left\| \mu - \sum_{j=1}^{\infty} \theta_j x_j \right\|^2.
\]

Results for four different cases are depicted in Figure 5. Again, in each case the risk is normalized by dividing by the risk of the infeasible optimal least squares estimator. It is seen from the figures that the MMA estimator habitually yields better estimates than the S-AIC and S-BIC estimators—these results are in accord with those observed by Hansen (2007). What is more striking is that the OPT estimator is found to be superior to the MMA estimator in a large region of the parameter space, and this superiority is most marked when \( n \) is large. This result is particularly encouraging given that the experiment has been performed under the same setting as Hansen’s (2007), where it was shown that the MMA estimator performs best. Results of the cases not depicted here have characteristics similar to those shown in Figure 5. Readers may refer to the online supplementary material for details.

**Example 3.** The third design is based on a model from Pearson and Timmermann (1994), who considered the predictability of excess returns for the Standard and Poor 500 index. The same model has been used to illustrate the consequences of ignoring pre-testing on forecasts by Danilov and Magnus (2004a). The model is expressed in the following equation:

\[
y_t = \beta_1 + \beta_2 \Pi_{t-2} + \beta_3 D\bar{I}_t - 1 + \beta_4 S\bar{P}R\bar{A}D_{t-1} + \gamma_1 YS\bar{P}_{t-1} + \gamma_2 D\bar{I}P_{t-1} + \gamma_3 P\bar{E}R_{t-1} + \gamma_4 D\bar{L}E\bar{A}D_{t-2} + \varepsilon_t, \quad (27)
\]

where \( y_t \) is excess returns, \( \Pi_{t-2} \) is annual inflation rate (lagged two periods), \( D\bar{I}_t - 1 \) is change in 3-month T-bill rate (lagged one period), \( S\bar{P}R\bar{A}D_{t-1} \) is credit spread (lagged one period), \( YS\bar{P}_{t-1} \) is dividend yield on SPS00 portfolio (lagged one period), \( D\bar{I}P_{t-1} \) is annual change in industrial production (lagged one period), \( P\bar{E}R_{t-1} \) is price-earnings ratio (lagged one period), and \( D\bar{L}E\bar{A}D_{t-2} \) is annual change in leading business cycle indicator (lagged two periods). The data contain 46 annual observations on each of the variables described above over the period 1956–2001. The data and their sources were given in the work of Danilov and Magnus (2004a). Specifically, Danilov and Magnus (2004a) were uncertain whether the last four regressors, namely, \( YS\bar{P}_{t-1}, D\bar{I}P_{t-1}, P\bar{E}R_{t-1}, \) and \( D\bar{L}E\bar{A}D_{t-2} \) should be included. The regressors \( \Pi_{t-2}, D\bar{I}_t - 1, S\bar{P}R\bar{A}D_{t-1} \) and the intercept are focus regressors that are required to be in the model. Danilov and Magnus (2004a) reported estimates from a (forward) stepwise model selection procedure which discarded all auxiliary regressors but \( YS\bar{P}_{t-1} \).

Alternative approaches could be the use of the OPT and MMA estimators described above. The OPT estimator takes average across \( 2^4 = 16 \) models. Again, for the MMA scheme, one need not distinguish between focus and auxiliary regressors, but must first order the regressors. Since the model contains eight regressors including the intercept term, there are

![Figure 5. Results for Example 2: risk comparisons under \( L^{(3)} \) loss.](image-url)
8!, = 40,320 possible ways to order the regressors. After ordering, the MMA scheme averages over eight models obtained by adding the eight regressors one at a time to the regression model. While in practice the regressors are ordered based on some preconceived notions of the investigator, for this discussion we consider all 40,320 possible ordering sequences to give a comprehensive picture of the performance of the estimator in all cases. To increase the realism of the simulation, the values of the dependent variable in each round of the simulations are obtained by drawing 46 random disturbances with replacements from the residuals of the least squares estimation of (27). Denote the lth such sample of disturbances as $e^*_l$, and values of the dependent variable in the lth sample are generated using $Y^*_l = H\Theta + e^*_l$. The experiment uses the least squares estimates of the coefficients in (27) as the true parameter vector $\Theta$. A total of 100 samples are drawn, and the OPT predictor $\hat{\mu}_f = H\hat{\Theta}_f$ and the MMA predictor $\hat{\mu}_{\text{mma}} = H\hat{\Theta}_{\text{mma}}$ are computed. There are altogether 40,320 $\hat{\mu}_{\text{mma}}$'s depending on the ordering of regressors. It should be noted that no estimators are exactly the same among these 40,320 $\hat{\mu}_{\text{mma}}$'s. For each estimator of $H\Theta$, the risk under the squared error loss is calculated.

The key findings are presented as follows. The risk of $\hat{\mu}_f$ is 0.0749, while the risk of $\hat{\mu}_{\text{mma}}$ ranges from a minimum of 0.0583 to a maximum of 0.0893, depending on the pattern in the density of the coefficients in (27) as the true parameter vector $\Theta$. From (Hjort and Claeskens 2003), we have

$$\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} L \equiv \left( \frac{\partial \mu}{\partial \delta} \right)' J_{00}^{-1} M + \omega' \theta - \hat{\theta}(D),$$

(30)

where $\sqrt{n}(\hat{\mu} - \mu)$ converges in distribution to $\sqrt{n}$. The ML estimator of $\theta$ is analogous to $\hat{\theta}$ in Section 2.

To study the choice of $c$, we first present some notations. Denote by $J_{\text{full}}$ the $(k + m) \times (k + m)$ information matrix of the full model evaluated at the null point $(\delta, \gamma)$. That is,

$$J_{\text{full}} = \text{var} \left( \begin{array}{cc} U(y) \\ V(y) \end{array} \right) = \left( \begin{array}{cc} J_{00} & J_{01} \\ J_{10} & J_{11} \end{array} \right),$$

with inverse

$$J_{\text{full}}^{-1} = \left( \begin{array}{cc} J_{00} & J_{01} \\ J_{10} & J_{11} \end{array} \right),$$

where $U(y) = \partial \log f(y, \delta, \gamma) / \partial \delta$ and $V(y) = \partial \log f(y, \delta, \gamma) / \partial \gamma$ are the score functions.

Also, let $\pi_S$ be the projection matrix mapping the vector $v = (v_1, \ldots, v_m)'$ to its subvector $v_S = v_S$ of $v_j$ with $j \in S$. Denote $K = J_{11}^{-1} = (J_{11} - \omega J_{01} J_{00}^{-1})^{-1}$, $K_S = (\pi_S K_{11}^{-1} \pi_S')^{-1}$, $H_S = K^{-1/2} \pi_S \pi_S K^{-1/2}$, and $J = J_{10}^{-1} \partial \mu / \partial \delta - \partial \mu / \partial \gamma$ with the partial derivatives evaluated at the null point $(\delta, \gamma)$. Further, we define $H_{\phi}$ as the null matrix of size $m \times m$, where $\phi$ is the empty set.

From (Hjort and Claeskens 2003), we have

$$\hat{\theta}(D) = K_{11}^{1/2} \left( \sum_S c(S) d_S H_S K^{-1/2} D \right. \left. \sim N_n(\theta, K) \right)$$

is the limiting variable of $D_n$ in distribution, and $M \sim N_k(0, J_{00})$ is independent of $D$.

If we denote $Z = K^{-1/2} D$, then $Z \sim N(a^*, I)$ with $a^* = K^{-1/2} \theta$, and we can write $\hat{\theta}(D) = K_{11}^{1/2} \left( \sum_S c(S) d_S H_S Z \right) \sim K^{1/2} \hat{a}^*(Z)$. Thus, $L = \left( \frac{\partial \hat{\mu}}{\partial \delta} \right)' J_{00}^{-1} M + \omega' K^{1/2} \hat{a}^* (Z) \sim \hat{a}^* (Z)$. From this analysis we can show that the asymptotic risk of $\hat{\mu}$ is

$$R_o(\hat{\mu}) = E(\Lambda^2) = \tau_0^2 + E \left[ \omega' K^{1/2} \hat{a}^* (Z) - \omega' K^{1/2} a^* \right]^2,$$

(31)

with $\tau_0^2 = \left( \frac{\partial \mu}{\partial \delta} \right)' J_{00}^{-1} \frac{\partial \mu}{\partial \delta}$. Assume (for the time being) that $\tau_0$, $\omega$, $K$, and $\hat{a}^*(Z)$ are known, and let the weight $c(S) d_S$ be a continuous function. Further, assume that the piecewise continuous partial derivatives of $c(S) d_S$ with respect to $z$ exist as do their expectations. Then, by a similar proof technique to that used for Theorem 1, we can derive the following unbiased estimator of the asymptotic risk of $\hat{\mu}$:

$$\hat{R}_o(\hat{\mu}) = \tau_0^2 - \omega' K \omega + \omega' K^{1/2} (\hat{a}^*(Z) - Z) (\hat{a}^*(Z) - Z)' K^{1/2} \omega + 2 \omega' K^{1/2} \frac{\partial \hat{a}^*}{\partial Z} K^{1/2} \omega.$$
Note that $\hat{R}_d(\hat{\mu})$ depends on $\tau_0^2, \omega, K,$ and $\hat{a}^*(Z)$, which are actually unknown, and dependent on $\delta$ and/or $Z$. Also, the first and second terms on the right side of Equation (32) are independent on the model. Thus, we suggest a criterion of weight selection that minimizes the following quantity:

$$\hat{R}_d(\hat{\mu}) = \left( \hat{a}^* \hat{K}^{1/2}(\hat{a}^*(Z_n) - Z_n) \right)^2 + 2\hat{a}^* \hat{K}^{1/2} \frac{\partial \hat{a}^*(Z_n)}{\partial Z} \hat{K}^{1/2},$$

(33)

which is obtained by neglecting the first two terms on the right side of (32), and replacing $J_{\text{full}}$ and $\delta$ in the same equation by their estimators $\hat{J}_{\text{full}}$ and $\hat{\delta}$, respectively, and $Z$ by the approximation $Z_n = \hat{K}^{-1/2} D_n$.

Consider the simplest special case where there are only the full and narrow models. Then (32) reduces to

$$\hat{R}_d(\hat{\mu}) = \left( c_{\text{full}} - 1 \right)^2 (\hat{a}^* \hat{K}^{1/2} Z_n)^2 + 2 c_{\text{full}} \hat{a}^* \hat{K} \hat{\omega}.$$

(34)

It is easily shown that (34) is minimized at

$$c_{\text{full}} = 1 - \frac{\hat{a}^* \hat{K} \hat{\omega}}{(\hat{a}^* \hat{K}^{1/2} Z_n)^2},$$

(35)

and thus

$$c_{\text{narrow}} = 1 - c_{\text{full}} = \frac{\hat{a}^* \hat{K} \hat{\omega}}{(\hat{a}^* \hat{K}^{1/2} Z_n)^2}.$$

(36)

These weights are very close to but not exactly as the weights chosen by (Hjort and Claeskens 2003). This slight difference in weights is due to the fact that Hjort and Claeskens’s (2003) criterion minimizes the asymptotic risk itself, whereas our criterion seeks weights that minimize the unbiased estimator of the risk. In fact, when $\theta$ is the only unknown quantity, the corresponding weights in the work of (Hjort and Claeskens 2003) may be obtained by taking the expectations of the numerators and denominators separately in the weights (35) and (36), by noting that $E(\omega^* K^{1/2} Z)^2 = \omega^* K \omega + (\omega^* K^{1/2} \alpha^*)^2$. Work in progress evaluates the empirical performance of the weight choice criterion based on $\hat{R}_d(\hat{\mu})$ in (33).

We end this section by noting that the above results developed for iid $y_i$’s can be extended to the case of regression models under some mild regularity conditions. Assume that $y_i$’s are generated from the density

$$f_{\text{true}}(y|x) = f(y|x, \delta, \gamma) = f(y|x, \delta, \gamma_0 + \theta/\sqrt{n}),$$

where $\delta$ typically comprises a $k \times 1$ vector of regression coefficients $\beta$ and a scalar parameter $\sigma$. The matrix

$$J_{\text{n.full}} = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial \log f(y_i|x_i, \delta, \gamma_0)/\partial \delta}{\partial \log f(y_i|x_i, \delta, \gamma_0)/\partial \gamma}$$

$$= \left( \begin{array}{cc} J_{\text{n.00}} & J_{\text{n.01}} \\ J_{\text{n.10}} & J_{\text{n.11}} \end{array} \right)$$

is assumed to converge to a suitable positive definite matrix $J_{\text{n.full}}$ as $n$ tends to infinity. The extension to the regression models is accomplished by replacing $\hat{J}_{\text{full}}$ in $\hat{R}_d(\hat{\mu})$ of Equation (33) with a suitable estimate of $J_{\text{n.full}}$.

6. CONCLUDING REMARKS

There has been a quickening of interest in frequentist model averaging in recent years. This article suggests a new approach to select model weights for a linear regression FMA estimator. The proposed estimator has been shown to be quite promising, and yields improved estimator performance over the estimators developed in the literature in a wide variety of circumstances. Among the known FMA estimators, Hansen’s (2007) MMA estimator has considerable appeal, but to implement this estimator the regressors must be ordered at the outset. One practical issue addressed in our investigation is how the various patterns of ordering regressors affect the finite sample performance of the MMA estimator. The simulation results presented here suggest that the way the regressors are ordered is indeed a major determinant of the finite sample behavior of the MMA estimator. For the experiments considered, the risks of the MMA estimators corresponding to different patterns of regressor ordering can differ markedly. The OPT estimator requires no such prior ordering of regressors, and is supported by both asymptotic as well as analytic finite sample justifications. Another feature of our analytical framework is that it nests other weights such as those based on S-AIC and S-BIC as special cases. Note also that our proposed criteria permit comparisons of different weighting schemes. While the bulk of our analysis emphasizes the normal linear regression model, a similar weight choice mechanism is also developed for model averaging in general likelihood models.

Admittedly, if model averaging is performed over the full set or a large subset of extended models, the computational burden quickly increases as $m$ increases. In this regard, the model screening prior to combining approach advocated by Yuan and Yang (2005), or the orthogonalization method developed recently by Magnus, Powell, and Prüfer (2010), may be desirable alternatives to direct computation. Recently, Hansen (2008) extended the idea of Mallows model averaging to forecast combinations. It would be interesting to further extend the OPT approach to an out-of-sample forecasting setting. We end by reiterating that although model combination captures the uncertainty inherent in model selection, from a statistical inference viewpoint, model combination itself does not guarantee that subsequent inference would be on sound footing; in order for post-model averaging to produce the “correct” inference, one has to work with distribution of the model average estimator. Studies of the distributional properties of the FMA estimators are of recent vintage. Readers are referred to the work of Pötscher (2006), who investigated the distributional properties of a special case of the FMA estimator discussed by Leung and Barron (2006). It remains a challenging endeavor to derive the full distribution of the OPT estimator discussed here.

APPENDIX: PROOFS OF THEOREMS

Proof of Theorem 1

The MSE of $\hat{\beta}_f$ may be written as

$$\text{MSE(}\hat{\beta}_f) = E((\hat{\beta}_f - \beta)^2) = \sigma^2(X'X)^{-1} + QE((W\hat{\theta} - \theta)^2)Q'$$

(A.1)
Equation (A.4) can be proven by noting that
\[ \hat{\Psi}(\theta, \hat{\sigma}^2) \] can be shown by Stein’s Lemma (Stein 1981) that
\[ E[\lambda_i(\hat{\theta}, \hat{\sigma}^2)(\hat{\theta} - \theta)\hat{\sigma}^2] = \sigma^2 E[\lambda_i(\hat{\theta}, \hat{\sigma}^2)\hat{\theta} - \theta, \hat{\sigma}^2/\hat{\sigma}^2]. \] 

Hence,
\begin{align*}
Q & = \sum_{i=1}^{N} E[\lambda_i(\hat{\theta}, \hat{\sigma}^2)(\hat{\theta} - \theta)\hat{\sigma}^2](W_i - I_m) \\
& = \sigma^2 Q(\theta, \hat{\sigma}^2) - \sigma^2 Q'.
\end{align*} 

Further,
\begin{align*}
E_{\hat{\sigma}^2}[\Psi(\hat{\theta}, \hat{\sigma}^2)] &= \sigma^2 E_{\hat{\sigma}^2}[\Psi(\hat{\theta}, \hat{\sigma}^2)].
\end{align*} 

Equation (A.4) can be proven by noting that \( (n - k - m)\hat{\sigma}^2/\sigma^2 \sim X^2(n - k - m) \), resulting in
\[ p(\theta) = \frac{(n - k - m)(n - k - m)/2}{2(n - k - m)^2/2\Gamma((n - k - m)/2)}(\sigma^2)^{(n - k - m)/2} \times \frac{1}{(n - k - m)/2 - 1} e^{-(n - k - m)/2}\tau(\sigma^2) \]
as the density function of \( \hat{\sigma}^2 \). Thus,
\[ E_{\hat{\sigma}^2}[\Psi(\hat{\theta}, \hat{\sigma}^2)] = \int_{0}^{\infty} \Psi(\hat{\theta}, \hat{\sigma}^2) \cdot C_0(\tau) \cdot r(\theta, \nu) e^{-(n - k - m)/2}\tau(\sigma^2) dt \]
and
\[ \theta' P_1 P_2 \theta = y'MZ^{-1} - S_j'(S_j'(MZ)^{-1}S_j^{-1}) - S_j'(MZ)^{-1} - Z'My. \]

By the definitions of \( P_i \) and \( \hat{\theta} \), we have
\[ \int_i \hat{\theta} + P_j \hat{\theta} = y'MZ^{-1}(S_j'(MZ)^{-1}S_j^{-1}) - S_j'(MZ)^{-1} - Z'My. \]

Combining (A.9), (A.11), (A.12), and (A.13), we obtain
\[ \int_i \hat{\theta} + P_j \hat{\theta} = y'MZ^{-1}(S_j'(MZ)^{-1}S_j^{-1}) - S_j'(MZ)^{-1} - Z'My. \]

It then follows that
\[ \lambda' \cdot (a, b, c) \cdot \hat{\lambda} \cdot (a, b, c) = \| \hat{\lambda} \cdot (a, b, c) - y \|^2 - (n - k - m)\hat{\sigma}^2. \]
By the definition of \( \hat{\sigma} \), we see that
\[
\lambda'(a, b, c) \hat{\sigma} = \lambda'(a, b, c)q - k, \tag{A.16}
\]
where \( q = (q_1, \ldots, q_N)' \). Now, let \( H_i \) be the regressor matrix in the \( \text{th} \) submodel, \( T_i = H_i' H_i^{-1} H_i' \), and \( A_i = I_n - T_i \). For any weight vector \( w \), define \( T(w) = \sum_{i=1}^N w_i T_i \) and \( A(w) = \sum_{i=1}^N w_i A_i \). We can define \( T(\lambda(a, b, c)) \) and \( A(\lambda(a, b, c)) \) in a similar way. Hence, it is clear that for \( i = 1, \ldots, N \),
\[
\hat{\mu}_i = T_i y,
\tag{A.17}
\]
and thus \( \hat{\mu}_f(\lambda(a, b, c)) = T(\lambda(a, b, c))y \). From (A.15) and (A.16), we obtain
\[
\begin{align*}
\hat{R}_n(\hat{\mu}_f(\lambda(a, b, c))) &= \| \hat{\mu}_f(\lambda(a, b, c)) - y \|^2 + 2\hat{\sigma}^2 \lambda'(a, b, c)q - n\hat{\sigma}^2 - 4(n/n)c\hat{\sigma}^2 \lambda'(a, b, c)\hat{\sigma}_a(n, b, c) \\
&= \| \hat{\mu}_f(\lambda(a, b, c)) - \mu \|^2 - 2\lambda(\hat{\sigma}(\lambda(a, b, c)) - \mu)^T \varepsilon + \| \varepsilon \|^2 \\
&+ 2\hat{\sigma}^2 \lambda'(a, b, c)q - n\hat{\sigma}^2 - 4(n/n)c\hat{\sigma}^2 \lambda'(a, b, c)\hat{\sigma}_a(n, b, c) \\
&= L_n(\lambda(a, b, c)) + 2\mu'A(\lambda(a, b, c))\varepsilon - 2\sigma^2 T(\lambda(a, b, c))\varepsilon \\
&+ 2\hat{\sigma}^2 \lambda'(a, b, c)q - (4/n)c\hat{\sigma}^2 \lambda'(a, b, c)\hat{\sigma}_a(n, b, c) \\
&- n\hat{\sigma}^2 + \| \varepsilon \|^2 \\
&= L_n(\lambda(a, b, c)) + t_n(a, b, c) - n\hat{\sigma}^2 + \| \varepsilon \|^2. \tag{A.18}
\end{align*}
\]
where the last two terms on the right side of (A.18) are unrelated to \( a, b, \) or \( c \), and
\[
t_n(a, b, c) = 2\mu'A(\lambda(a, b, c))\varepsilon - 2\sigma^2 T(\lambda(a, b, c))\varepsilon \\
+ 2\hat{\sigma}^2 \lambda'(a, b, c)q - (4/n)c\hat{\sigma}^2 \lambda'(a, b, c)\hat{\sigma}_a(n, b, c).
\]
Therefore, we can write
\[
(\hat{\sigma}, \hat{\lambda}, \hat{\varepsilon}) = \arg \min_{(\lambda, \sigma, \varepsilon) \in D_0} \left\{ -L_n(\lambda(a, b, c)) + t_n(a, b, c) \right\}. \tag{A.19}
\]
Consequently,
\[
\inf_{(\lambda, \sigma, \varepsilon) \in D_0} \left\{ -L_n(\lambda(a, b, c)) + t_n(a, b, c) \right\} = L_n(\lambda(\hat{\sigma}, \hat{\lambda}, \hat{\varepsilon})) + t_n(\hat{\sigma}, \hat{\lambda}, \hat{\varepsilon}). \tag{A.20}
\]
On the other hand, by noting that
\[
R_n(w) = EL_n(w) = E\| T(w)y - \mu \|^2 \\
= \| T(w)y - \mu \|^2 + 2\sigma^2 \text{tr}[T^2(w)] \\
= \| T(w)y - \mu \|^2 + 2\| T(w) \|^2 \\
- 2\text{tr}[T^2(w)] \\
= L_n(w) + 2\mu'A(\lambda(\sigma, \varepsilon))w + 2\sigma^2 \text{tr}[T^2(w)] \\
= L_n(w) + 2\mu'A(\lambda(\sigma, \varepsilon))w + 2\sigma^2 \| T(w) \|^2 \\
- 2\| T(w) \|^2 - 2\| T(\lambda(\sigma, \varepsilon))w \|^2 + 2\| T(\lambda(\sigma, \varepsilon))w \|^2 \\
\]
we have
\[
L_n(w) = R_n(w) + u_n(w), \tag{A.21}
\]
where \( u_n(w) = \| T(\lambda(\sigma, \varepsilon))w \|^2 - 2\sigma^2 \text{tr}[T^2(w)] - 2\mu'A(\lambda(\sigma, \varepsilon))w \).

Also, by the definition of infimum, there exist a series of nonnegative \( \bar{\sigma}_n \) and sets \( (a_n, b_n, c_n) \in D_0 \) such that \( \bar{\sigma}_n \to 0 \) when \( n \to \infty \), and
\[
\inf_{(a, b, c) \in D_0} L_n(\lambda(a, b, c)) = L_n(\lambda(a_n, b_n, c_n)) - \bar{\sigma}_n. \tag{A.23}
\]
Now, from (A.20), (A.22), and (A.23), it can be shown that for any \( \delta > 0 \),
\[
\begin{align*}
\Pr \left\{ \inf_{(a, b, c) \in D_0} \frac{L_n(\lambda(a, b, c))}{L_n(\lambda(\hat{\sigma}, \hat{\lambda}, \hat{\varepsilon}))} - 1 > \delta \right\} \\
\leq 2\Pr \left\{ \sup_{(a, b, c) \in D_0} \frac{t_n(a, b, c)}{\bar{\sigma}_n} \leq \frac{1}{1 + \inf_{w \in \mathcal{W}} u_n(w)/\bar{\sigma}_n} \geq \frac{\delta}{3} \right\} \\
+ 3\Pr \left\{ \inf_{w \in \mathcal{W}} u_n(w)/\bar{\sigma}_n \geq 1 \right\}. \tag{A.24}
\end{align*}
\]
The detailed proof of (A.24) is available in the online supplementary material. Hence, to demonstrate the theorem, it suffices to prove that, as \( n \to \infty \),
\[
\sup_{(a, b, c) \in D_0} \frac{t_n(a, b, c)/\bar{\sigma}_n}{\bar{\sigma}_n} \to 0, \tag{A.25}
\]
and
\[
\bar{\sigma}_n/\bar{\sigma}_n \to 0. \tag{A.27}
\]
Noting that \( \bar{\sigma}_n \to 0 \), the convergence described in Equation (A.27) is obvious from condition (22). By the same condition, together with Chebyshev's inequality, theorem 2 of the article by Whittle (1960), and the fact that \( \varepsilon \sim N(0, \sigma^2 I_0) \), it can be shown that (see the online supplementary material for detailed proofs)
\[
\sup_{w \in \mathcal{W}} \frac{|\mu'(A(w))\varepsilon|}{\bar{\sigma}_n} \to 0, \tag{A.28}
\]
\[
\sup_{w \in \mathcal{W}} |\varepsilon' T(w)\varepsilon - \sigma^2 w' w|/\bar{\sigma}_n \to 0, \tag{A.29}
\]
\[
\sup_{w \in \mathcal{W}} |\mu'(A(w))T(w)\varepsilon|/\bar{\sigma}_n \to 0, \tag{A.30}
\]
and
\[
\sup_{w \in \mathcal{W}} \| T(w) \|^2 - 2\sigma^2 \text{tr}[T^2(w)]/\bar{\sigma}_n \to 0. \tag{A.31}
\]
So, by (A.30), (A.31), and the definition of \( u_n(w) \), we see that (A.26) is true. If we let
\[
t_n(w, c) = 2\mu'A(\lambda(\sigma, \varepsilon))w - 2\sigma^2 T(w)\varepsilon + 2\hat{\sigma}^2 w' q - (4/n)c\hat{\sigma}^2 w' \hat{\sigma}_a(n, b, c) \\
\]
then in order for (A.25) to hold, we need only to prove
\[
\sup_{w \in \mathcal{W}, -\varepsilon \leq c \leq 0} \frac{|\mu'(A(w))w|}{\bar{\sigma}_n} \to 0. \tag{A.33}
\]
In light of (A.28) and (A.29), to complete the proof, we need only to consider the last term on the right side of (A.32). That is, to prove that (A.25) is true, it suffices to show
\[
\sup_{w \in \mathcal{W}, -\varepsilon \leq c \leq 0} \frac{|c\hat{\sigma}^2 w' \hat{\sigma}_a(n, b, c)w|}{\bar{\sigma}_n} \to 0. \tag{A.34}
\]
From (A.14) and \( \hat{\sigma}^2 = \frac{(n-k-m)\hat{\sigma}^2 + \hat{\theta}'P\hat{\theta}}{n} \), we see that for any \( 1 \leq i,j \leq N \),
\[
|\hat{\sigma}^2 g_{ij}/n|/n \leq \frac{|\hat{\theta}'P\hat{\theta} - \hat{\theta}'P\hat{\theta}|}{(n-k-m)\hat{\sigma}^2 + \hat{\theta}'P\hat{\theta}} \leq \frac{|\hat{\theta}'P\hat{\theta} - \hat{\theta}'P\hat{\theta}|}{n-k-m}
\]
where \( S(\cdot) \) denotes the largest singular value of a matrix. The last inequality in (A.35) results from \( S(U_1U_2) \leq S(U_1)S(U_2) \), and \( S(U_1) + S(U_2) \leq S(U_1) + S(U_2) \) for any \( n \times n \) matrices \( U_1 \) and \( U_2 \) (see Li 1987), while the last equality in (A.35) is obtained from condition (21).

The proof is completed by noting that condition (22) and (A.35) imply (A.34).

SUPPLEMENTARY MATERIALS

Proofs and Results: The supplemental materials contain detailed proofs and additional simulation results as follows (SupplementaryMaterial.pdf):

- Derivation of \( \Psi(\hat{\theta}, \hat{\sigma}^2) \) in Theorem 1
- Numerical comparison of \( \hat{\sigma}^2 \Psi_1(\hat{\theta}, \hat{\sigma}^2) \) and \( \Psi(\hat{\theta}, \hat{\sigma}^2) \)
- Proof of (A.24)
- Proof of (A.28)
- Proof of (A.29)
- Proofs of (A.30) and (A.31)
- Proof of the result on the upper bound of the simple example in Section 3.3
- General bounds of the sum \( \sum_{c \in \mathcal{U}} \lambda_c(a, b, c) \) in Section 3.3
- Proofs of the results related to the simple example in Section 3.3
- Other examples in which condition (22) is satisfied
- Further results for simulation Example 1 in Section 4
- Further results for simulation Example 2 in Section 4.

REFERENCES


Yang, Y. (2003), “Regression With Multiple Candidate Models: Selecting or Mixing?” Statistica Sinica, 13, 783–809. [1053,1058]