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Regression Analysis

Generalized Liu Type Estimators Under Zellner’s Balanced Loss Function

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In regression analysis, ridge regression estimators and Liu type estimators are often used to overcome the problem of multicollinearity. These estimators have been evaluated using the risk under quadratic loss criterion, which places sole emphasis on estimators’ precision. The traditional mean square error (MSE) as the measure of efficiency of an estimator only takes the error of estimation into account. In 1994, Zellner proposed a balanced loss function. Here, we consider the balanced loss function which incorporates a measure for the goodness of fit of the model as well as estimation precision. We also examine the risk performance of the feasible generalized Liu estimator and feasible almost unbiased generalized Liu estimator when the balanced loss function is used.

Keywords Balanced loss; Collinearity; Liu estimator; Ridge regression; Risk.

Mathematics Subject Classification 62J05; 62J07.

1. Introduction

We consider the estimation of the parameters of the model

$$y = Z\gamma + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I)$$  (1.1)

where y and ε are \((n \times 1)\); \(\gamma\) is \((p \times 1)\); Z is \((n \times p)\), non stochastic and of rank \(p(<n)\). The least squares estimator \(\hat{\gamma} = (Z'Z)^{-1}Z'y\) is commonly used to estimate \(\gamma\).
The least squares estimator performs poorly in the presence of collinearity. The impact of collinearity on least squares estimators can be very serious if the primary interest is in the regression coefficients.

The least squares estimators of regression coefficients are still unbiased but the covariance matrix of \( \hat{\beta} \) may contain some large values, as the eigenvalues of \( Z'Z \) will differ considerably in magnitude with some roots being close to 0 in the face of multicollinear data. To solve this problem, Hoerl and Kennard (1970) proposed ridge regression estimator

\[
\hat{\beta}_R = (Z'Z + kI)^{-1}Z'y
\]

and it has become the most common method to overcome the weakness of least squares estimator.

First rewrite model (1.1) in canonical form.

\[
y = X\beta + \epsilon
\]

where \( X = ZT \), and \( \beta = T'\gamma \), and \( T \) is the orthogonal matrix whose columns constitute the eigenvectors of \( Z'Z \). Then \( X'X = T'Z'ZT = \Lambda \), where \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p) \) and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \) are the ordered eigenvalues of \( Z'Z \). For model (1.2), the ordinary least squares estimator (OLSE) of \( \beta \) is

\[
\hat{\beta} = \Lambda^{-1}X'y
\]

and the covariance matrix of \( \hat{\beta} \) is

\[
\text{Var}(\hat{\beta}) = \sigma^2 \Lambda^{-1}.
\]

To overcome the collinearity problem, Liu (1993) proposed the following estimator,

\[
\tilde{\beta}_D = (\Lambda + I)^{-1}(X'y + D\hat{\beta})
\]

\[
= (\Lambda + I)^{-1}(\Lambda + D)\hat{\beta}
\]

\[
= [I - (\Lambda + I)^{-1}(I - D)]\hat{\beta}
\]

where \( D = \text{diag}(d_1, \ldots, d_p) \) and the \( d_i \)'s are biasing parameters. The estimator \( \tilde{\beta}_D \) is called the generalized Liu estimator (GLE) by Akdeniz and Kaçiranlar (1995). Akdeniz and Kaçiranlar (1995) derived the almost unbiased generalized Liu estimator and examined an exact unbiased estimator of the bias and mean squared error (MSE) of the feasible generalized Liu estimator. They showed that choosing

\[
d_{i(\text{opt})} = \frac{\lambda_i\beta_i^2 - \sigma^2}{\lambda_i^2 + \sigma^2}
\]

minimizes the MSE of \( \tilde{\beta}_i \), where \( \beta_i \) and \( \tilde{\beta}_i \) are the \( i \)th element of \( \beta \) and \( \tilde{\beta}_D \), respectively. Thus, we have the minimal \( \text{mse}(\tilde{\beta}_i) = \frac{\sigma^2 \beta_i^2}{\lambda_i^2 + \sigma^2} \) (see Akdeniz et al., 1999). The properties of \( \tilde{\beta}_D \) were studied by Akdeniz (2001) and Akdeniz and Namba (2003). The feasible generalized Liu estimator (FGLE) is just the operational counterpart to this optimal Liu estimator, namely

\[
\tilde{\beta}^* = [I - (\Lambda + I)^{-1}(I - D)]\hat{\beta}
\]
where $\hat{D} = diag(\hat{d}_1, \ldots, \hat{d}_p)$, $\hat{d}_i = \frac{i(\hat{\beta}^2_i - \hat{\sigma}^2)}{\hat{\lambda}_i \hat{\beta}^2_i + \hat{\sigma}^2}$, $\hat{\sigma}^2 = (y - X\hat{\beta})'(y - X\hat{\beta})/v$, $v = n - p$.

Clearly, the FGLE does not have any optimal MSE property as a result of the substitution of the optimal $d_i$ by $\hat{d}_i$. Following Kadiyala (1984), the bias corrected generalized Liu estimator of $\beta$ is given by

$$\tilde{\beta}_{BC} = \tilde{\beta}_D + \frac{1}{(I - D)^{-1} \Lambda D}.$$ (1.8)

If we replace the $\beta$ by the biased estimator $\tilde{\beta}_D$, we have the almost unbiased generalized Liu estimator (AUGLE), $\tilde{\beta}_o$, is given by

$$\tilde{\beta}_o = [\tilde{\beta}_D + (\Lambda + I)^{-1} (I - D) \tilde{\beta}_D]$$ $$= [I + (\Lambda + I)^{-1} (I - D)] \tilde{\beta}_D$$ $$= [I - (\Lambda + I)^{-1} (I - D)^2] \tilde{\beta}.$$ (1.9)

A feasible version of $\tilde{\beta}_o$, as suggested by Akdeniz (2004), is to replace by $D$ by $\hat{D}$ in (1.9) yielding

$$\tilde{\beta}_{oo} = [I - (\Lambda + I)^{-2} (I - \hat{D})^2] \hat{\beta},$$ (1.10)

the almost unbiased feasible generalized Liu estimator (AUFGL).  

It is interesting to note that much of the studies on the biased estimators use the weak mean square error criterion, or equivalently, the quadratic loss function as a measure of estimators’ performance. This article extends these studies by the choice of the loss function used to decide on a preferred estimator of $\beta$. The traditional procedure used is to consider a quadratic loss function, which concentrates purely on parameter estimation decision. However, not much is known about robustness of the estimators’ risk properties to departures from quadratic loss, although see, for example, Ohtani (1995), Wan (1999), Akdeniz (2004), and Akdeniz and Namba (2003) who considered the risk behavior of the feasible generalized ridge regression estimator (FGRE), FGLE, almost unbiased feasible generalized ridge regression estimator (AUFGRE), and AUFGL using the asymmetric LINEX loss functions. Both the quadratic and LINEX loss functions place sole emphasis on the precision of an estimator and ignore issues such as goodness of fit. In regression analysis we are often interested in using an estimator which is precise and which simultaneously provides a model with good fit. Here we follow Zellner (1994) and allow the loss function to incorporate a measure of the “goodness of fit” of the model as well as the precision of estimation. In general, estimation precision and the overall “goodness of fit” of a model may be different issues especially in the case of biased estimators.

Zellner (1994) proposed the balanced loss function as a means of incorporating both goodness of fit and precision of estimation in the evaluation of an estimator. A generalization of Zellner’s balanced loss function is proposed by Gruber (2004). A general form of the balanced loss for the above setup is

$$L_w(\theta, \tilde{\theta}) = w(y - X\tilde{\theta})'(y - X\tilde{\theta}) + (1 - w)(\theta - \tilde{\theta})'(\theta - \tilde{\theta})$$ (1.11)

when $0 \leq w \leq 1$ is the relative weight given to the goodness-of-fit portion of the loss and $1 - w$ is the relative weight given to the precision of estimation portion.
Clearly, when $w = 0$, the BLF reduces to the quadratic loss function. Also, as the sum of the squared residuals is used as a measure of goodness of fit in (1.11), the OLSE is optimal with respect to the BLF when $w = 1$. Wan (2002) derived and numerically evaluated the risks of the FGRE and AUFGRE under the balanced loss structure given in (1.11).

This article extends the literature on the Liu type estimators by considering the balanced loss function as a basis of measuring the performance of the feasible generalized Liu and almost unbiased feasible generalized Liu estimators. We derive and numerically evaluate the risks of the FGLE and AUFGLE under the loss structure given in (1.11). Next, we find that both of these estimators continue to improve over the ordinary least squares estimator in the case of ill-conditioned data, even if a relatively heavy weight is given to goodness of fit in the balanced loss function.

2. Risk of the Estimators

**Theorem 2.1.** Under the balanced loss function, the scaled risks of the FGLE and AUFGLE are given by

$$R(\tilde{\beta}^*)/\sigma^2 = w \left[ v + \sum_{i=1}^{p} \lambda_i H(2,2) \right]$$

$$+ (1 - w) \left[ \sum_{i=1}^{p} \left( \frac{1}{\lambda_i} - 2H(1,2) + H(2,2) + 2\beta_i H(1,1) \right) \right]$$

(2.1)

and

$$R(\tilde{\beta}^{oo})/\sigma^2 = w \left[ v + \sum_{i=1}^{p} \lambda_i H(4,2) \right]$$

$$+ (1 - w) \left[ \sum_{i=1}^{p} \left( \frac{1}{\lambda_i} - 2H(2,2) + H(4,2) + 2\beta_i H(2,1) \right) \right]$$

(2.2)

respectively, where

$$H(a, b) = E \left[ \left( \frac{1 - \hat{d}_i}{1 + \lambda_i} \right)^a \hat{\beta}_i^b \right] / \sigma^2 = \sum_{q=0}^{\infty} \frac{\Gamma(b+q+1)}{q! \Gamma(q+3/2) \Gamma(q+1/2)} \exp(-2 \theta_i^2/\sigma^2) \theta_i^{2q-b+1}$$

$$\times 2^{b+1-q} \beta_i^b \int_0^1 \frac{f^q(1-f)^{a+q-1}}{[1+f(v-1)]^a} df/\sigma^2$$

(2.3)

if $b$ is odd, or

$$H(a, b) = E \left[ \left( \frac{1 - \hat{d}_i}{1 + \lambda_i} \right)^a \hat{\beta}_i^b \right] / \sigma^2 = \sum_{q=0}^{\infty} \frac{\Gamma(b+q+1)}{q! \Gamma(q+1/2) \Gamma(q+1/2)} \exp(-2 \theta_i^2/\sigma^2) \theta_i^{2q-b}$$

$$\times 2^{b+q+1-q} \beta_i^b \int_0^1 \frac{f^q(1-f)^{a+q-1}}{[1+f(v-1)]^a} df/\sigma^2$$

(2.4)

if $b$ is even.

**Proof.** See Appendix A.
Table 1

<table>
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<tr>
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<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
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<td>0.0001</td>
<td>$1 \times 10^{-9}$</td>
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</table>

### 3. Numerical Evaluations

In this section we numerically evaluate the risks of the FGLE and AUFGLE. As in Hill and Judge (1987) and Wan (2002), we consider varying degrees of multicollinearity by specifying values for the eigenvalues of $Z'Z$. The configurations used are given in Table 1. In Case 1, the data are orthonormal. In Case 2, the eigenvalues decline linearly with one near-exact linear dependency, while there are several near-exact linear dependencies in Case 3. In Case 4, the roots are in three groupings with one having very small roots. We consider the following parameter values: $v = 15, 25$ and $50$, and $w = 0, 0.2, 0.5, 0.7, 0.9, \text{ and } 1.0$. Following Hill and Judge (1987) and Wan (2002), we choose the coefficient vector by setting $\beta = Lc$, where $L$ is the length of the parameter vector and $c$ is a $(10 \times 1)$ vector whose values are $(1/10)^{1/2}$ so that $c'c = 1$. We use the NAG subroutine D01AHF to evaluate the

![Figure 1](image_url). Relative risks of estimators under balanced loss [Case 1, $v = 15, w = 0.0$].
integrals and the subroutines GAMMLN and FACTLN from Press et al. (1992) to calculate the gamma functions and the factorials in (2.3) and (2.4).

For ease of comparisons, the risks of the OLSE have been scaled to one in each case. Figures 1–3 give the results for Case 1, i.e., orthonormal regressors. In Fig. 1, $w = 0$ and the balanced loss function reduces to quadratic loss. For this case, neither the FGLE nor the AUFGLE uniformly dominate one another or the OLSE. Of the three estimators, the FGLE has the smallest risk when $L$ is small; for moderate to large $L$ values, the OLSE dominates both the FGLE and the AUFGLE. There is no region in the parameter space where the AUFGLE has the best performance in

Figure 2. Relative risks of estimators under balanced loss [Case 1, $v = 15$, $w = 0.7$].

Figure 3. Relative risks of estimators under balanced loss [Case 1, $v = 15$, $w = 0.9$].
terms of risk, but there is always a region where it has the worst risk performance among the three estimators considered. Other things being equal, increasing $w$ has the effect of increasing the relative risks of both the FGLE and AUFGLE (see Fig. 2). For a sufficiently large value of $w$ (say, $w > 0.7$), both the FGLE and AUFGLE are dominated by the OLSE estimator uniformly over the entire region of the parameter space (see Fig. 3). As discussed in Wan (2002), this result is perhaps not surprising because the OLSE minimizes the sum of squared residuals and therefore when $w = 1$, it dominates other estimators for which the minimization of the sum of squared residuals is not a criterion. Other works, including Wan (1994), have also cited instances in which the OLSE becomes the dominating estimator when a

**Figure 4.** Relative risks of estimators under balanced loss [Case 3, $v = 15$, $w = 0.2$].

**Figure 5.** Relative risks of estimators under balanced loss [Case 4, $v = 15$, $w = 0.9$].
sufficiently large weight is given to goodness of fit in the balanced loss function. When $w$ is large, the FGLE is the worst of the three estimators in terms of risk under balanced loss. Some of these results continue to hold for Cases 2–4, where the design matrices are more ill-conditioned. The only exceptions are (i) for situations where the FGLE and AUFGLE have smaller risks than the OLSE in some parts of the parameter space, these estimators tend to improve over the OLSE over a much larger region than in Case 1 and (ii) it can happen that even when 90% of the weight is given to goodness of fit, the FGLE and AUFGLE still have better risk performance than the OLSE over some regions of the parameter space (see, for example, Figs. 4 and 5). When $w = 1$, the OLSE becomes the dominating estimator in all cases considered without exception. Generally speaking, other things being equal, the larger $\lambda_i/\lambda_i$ is, the closer the risks of the FGLE and AUFGLE are to the risks of the OLSE. Overall, the results observed are qualitatively similar to those given in Wan (2002), where the analysis is in terms of the almost unbiased generalized ridge estimator.

4. Conclusions

This article gives some numerical results by considering the balanced loss function as a basis of measuring the risk performance of the FGLE and AUFGLE.

It is well known that the OLSE minimizes the sum of squared residuals and hence dominates both the FGLE and AUFGLE when $w = 1$. When the data are multicollinear, the Liu type estimators are still recommended even if “goodness of fit” is considered to be of much greater importance than “estimation precision” by the researcher.

Appendix A

Under the assumptions of Sec. 1, the risk of any estimator $\tilde{\beta}$ of $\beta$ under the BLF can be written as

$$R(\tilde{\beta})/\sigma^2 = \sigma^{-2}wE[(y - X\tilde{\beta})'(y - X\tilde{\beta})] + \sigma^{-2}(1 - w)E[(\tilde{\beta} - \beta)'(\tilde{\beta} - \beta)]$$

$$= \sigma^{-2}wE[(y - X\tilde{\beta}) + X(\tilde{\beta} - \tilde{\beta})]'[(y - X\tilde{\beta}) + X(\tilde{\beta} - \tilde{\beta})]$$

$$+ \sigma^{-2}(1 - w)E[(\tilde{\beta} - \beta)'(\tilde{\beta} - \beta)]$$

$$= \sigma^{-2}wE[e'e + (\tilde{\beta} - \tilde{\beta})'X(\tilde{\beta} - \tilde{\beta})] + \sigma^{-2}(1 - w)E[(\tilde{\beta} - \beta)'(\tilde{\beta} - \tilde{\beta})]$$

$$= \sigma^{-2}wE\left[e'e + \sum_{i=1}^{p}\lambda_i(\tilde{\beta}_i - \hat{\beta}_i)^2\right] + \sigma^{-2}(1 - w)\left[\sum_{i=1}^{p}E(\tilde{\beta}_i^2) - 2\beta_iE(\tilde{\beta}_i) + \beta_i^2\right]$$

(A.1)

Now from (1.7) and (1.10), we get

$$\tilde{\beta}_i^* = \hat{\beta}_i - \frac{1 - \tilde{d}_i}{1 + \lambda_i} \tilde{\beta}_i,$$

(A.2)

$$\tilde{\beta}_i^{**} = \hat{\beta}_i - \left(\frac{1 - \tilde{d}_i}{1 + \lambda_i}\right)^2 \tilde{\beta}_i.$$

(A.3)
Furthermore,

\[ E(\hat{\beta}_i) = \sigma \theta_i / \sqrt{\lambda_i} \]  \hspace{1cm} (A.4)

and

\[ E(\hat{\beta}_i^2) = \beta_i^2 + \sigma^2 / \lambda_i , \]  \hspace{1cm} (A.5)

where \( \theta_i = \sqrt{\lambda_i} \beta_i / \sigma \). Hence, we obtain \( E(\hat{\beta}_i^*) = \sigma \theta_i / \sqrt{\lambda_i} - E(1 - \frac{\hat{d}_i}{1 + \lambda_i \beta_i}) \). Now, to establish the explicit formula of \( H(a, b) \), let us write

\[ E\left[ \left( 1 - \frac{\hat{d}_i}{1 + \lambda_i} \right)^a \hat{\beta}_i^b \right] = \sigma^2 H(a, b). \]  \hspace{1cm} (A.6)

Hence, we have

\[ E(\tilde{\beta}_i^*) = \sigma \theta_i / \sqrt{\lambda_i} - \sigma^2 H(1, 1) \]  \hspace{1cm} (A.7)

\[ E(\tilde{\beta}_i^{**}) = \sigma \theta_i / \sqrt{\lambda_i} - \sigma^2 H(2, 1) \]  \hspace{1cm} (A.8)

\[ E(\tilde{\beta}_i^{*2}) = \sigma^2 \theta_i^2 / \lambda_i + \sigma^2 / \lambda_i + \sigma^2 H(2, 2) - 2\sigma^2 H(1, 2) \]  \hspace{1cm} (A.9)

\[ E(\tilde{\beta}_i^{**2}) = \sigma^2 \theta_i^2 / \lambda_i + \sigma^2 / \lambda_i + \sigma^2 H(4, 2) - 2\sigma^2 H(2, 2) \]  \hspace{1cm} (A.10)

\[ E(\tilde{\beta}_i - \tilde{\beta}_i^*)^2 = \sigma^2 H(2, 2) \]  \hspace{1cm} (A.11)

\[ E(\tilde{\beta}_i - \tilde{\beta}_i^{**})^2 = \sigma^2 H(4, 2) \]  \hspace{1cm} (A.12)

Substituting (A.4)–(A.12) in (A.1) and recognizing that \( E(e' e / \sigma^2) = \nu \) yield the expressions given in (2.1) and (2.2), where

\[
R(\tilde{\beta}^*) / \sigma^2 = w E\left[ e' e + \sum_{i=1}^{p} \lambda_i (\tilde{\beta}_i - \tilde{\beta}_i^*)^2 \right] / \sigma^2
\]

\[ + (1 - w) E\left[ \sum_{i=1}^{p} \{ E(\tilde{\beta}_i^{**2}) - 2 \beta_i E(\tilde{\beta}_i^*) + \beta_i^2 \} \right] / \sigma^2
\]

\[ = w \left[ \nu + \sum_{i=1}^{p} \hat{\lambda}_i H(2, 2) \right]
\]

\[ + (1 - w) \left[ \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i} - 2H(1, 2) + H(2, 2) + 2\beta_i H(1, 1) \right\} \right]
\]

and

\[
R(\tilde{\beta}^{**}) / \sigma^2 = w E\left[ e' e + \sum_{i=1}^{p} \lambda_i (\tilde{\beta}_i - \tilde{\beta}_i^{**})^2 \right] / \sigma^2
\]

\[ + (1 - w) E\left[ \sum_{i=1}^{p} \{ E(\tilde{\beta}_i^{**2}) - 2 \beta_i E(\tilde{\beta}_i^{**}) + \beta_i^2 \} \right] / \sigma^2
\]
where \( z_i = \sqrt{\frac{\lambda_i}{\sigma}} \sim N(\theta_i, 1) \) and \( V = \nu \hat{\sigma}^2 / \sigma^2 \sim \chi^2_\nu \). Hence,

\[
E \left\{ \left( \frac{V/v}{z_i^2 + V/v} \right)^{\beta_i b_i / \theta_i} \right\} = \int_0^\infty \int_{-\infty}^\infty \frac{(v)^a}{(z_i^2 + v)^a} z_i^b \beta_i^b \theta_i^{-b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_i - \theta_i)^2} \frac{1}{2^{\nu/2} \Gamma(\nu/2)} V^{\nu/2-1} e^{-V/2} dz_i dV.
\]

(A.13)

Noting that

\[
\exp \left( -\frac{1}{2}(z_i - \theta_i)^2 \right) = \exp(-\frac{z_i^2 + \theta_i^2}{2}) \sum_{l=0}^{\infty} \frac{z_i^l \theta_i^l}{l!},
\]

we can write (A.13) as

\[
= \int_0^\infty \int_{-\infty}^\infty \sum_{l=0}^{\infty} C_l z_i^{b+l} V^{a+\frac{1}{2}l-1} \frac{1}{(z_i^2 + V)^a} \exp \left( -\frac{z_i^2 + V}{2} \right) dz_i dV,
\]

(A.14)

where \( C_l = \nu^{-a} (2\pi)^{-1/2} [2^{\nu/2} \Gamma(\nu/2)]^{-1} \exp(-\theta_i^2/2) \beta_i^b \theta_i^{-b} / l! \).

Now, \( b \) is odd, the integral with respect to \( z_i \) in (A.14) is zero if \( l \) is even. So we consider only the case of \( l = 2q + 1, \ q = 1, 2, \ldots, \infty \). If we use the transformation \( W = z_i^2 \), then we get

\[
= \int_0^\infty \int_0^\infty 2 \sum_{q=0}^{\infty} C_q W^{q+\frac{1}{2}} V^{a+\frac{1}{2}} \frac{1}{(W + V)^a} \exp \left( -\frac{W + V}{2} \right) \frac{1}{2W^{1/2}} dW dV,
\]

\[
= \int_0^\infty \int_0^\infty \sum_{q=0}^{\infty} C_q W^{q+\frac{1}{2}} V^{a+\frac{1}{2}} \frac{1}{(W + V)^a} \exp \left( -\frac{W + V}{2} \right) dW dV.
\]

(A.15)

Making use the change of variables

\[
u = (V + W)/2 \quad \text{and} \quad f = W/(V + W)
\]
(A.15) reduces to
\[
\sum_{q=0}^{\infty} C_q \int_0^1 \int_0^\infty \frac{(2uf)^{q+b/2}[2u(1-f)]^{a-1+v/2}}{[2uf + 2u(1-f)]^a} \exp(-u) 4u \, du \, df \\
= \sum_{q=0}^{\infty} C_q^* \int_0^1 \int_0^\infty u^{(q+1+\frac{b}{2})-1} \exp(-u) \frac{f^{q+b/2}(1-f)^{a-1+v/2}}{[1+f(v-1)]^a} \, du \, df
\]
(A.16)

where \( C_q^* = C_q^{1/2} 2^{q+b/2+1} \). Note that
\[
\int_0^\infty u^{(q+\frac{b}{2}+1)-1} \exp(-u) du = \Gamma(q + \frac{b}{2} + 1)
\]
we obtain (A.17);
\[
\sum_{q=0}^{\infty} C_q^* \Gamma(q + \frac{b}{2} + 1) \int_0^\infty f^{q+b/2}(1-f)^{a-1+v/2} \left[1+f(v-1)\right]^a \, df.
\]
Using the duplication formula, \( \pi^2 (2q+1)! = q! 2^{2q+1} \Gamma(q + \frac{3}{2}) \), \( C_q^* \) reduces to
\[
C_q^* = \frac{\exp(-\theta_i^2/2) \theta_i^{2q-b+1}}{\Gamma(q + \frac{3}{2}) q! \Gamma(\frac{1}{2})} 2^{\frac{1}{2}-q} \theta_i^{b-q}.
\]
(A.18)

Substituting (A.18) in (A.17), we obtain the required expression in (2.3). Let \( b \) be even and the integral with respect to \( z_i \) in (A.14) is zero if \( l \) is odd. So we consider only the case of \( l = 2q, q = 1, 2, \ldots, \infty \). Noting that, \( \pi^2 (2q)! = q! 2^{2q} \Gamma(q + 1/2) \), we obtain (2.4).

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References


