ON THE BIAS AND MEAN SQUARE ERROR OF THE LEAST SQUARE ESTIMATOR
IN A REGRESSION MODEL WITH TWO INEQUALITY CONSTRAINTS
AND MULTIVARIATE T ERROR TERMS

Alan T.K. Wan

Department of Econometrics,
University of New South Wales,
Sydney, N.S.W. 2052,
Australia

Key Words and Phrases: inequality restrictions; linear regression;
multivariate t distribution

ABSTRACT

We derive and numerically evaluate the bias and mean square error
of the inequality constrained least squares estimator in a model with
two inequality constraints and multivariate t error terms. Our results
suggest that qualitatively, the estimator properties found for models
with normal errors carry over to the case of multivariate t errors.

1. INTRODUCTION

The properties of the inequality constrained least squares (ICLS)
estimator have been widely examined in recent years. Much of the
existing literature has focused on the case where the coefficients in

It is interesting to note that with the exception of Ohtani (1991), all these studies assume that the error terms are normally distributed. One particular form of non-normal distribution which has received considerable attention in the literature is the multivariate t distribution. See, for instance, Zellner (1976), Sutradhar and Ali (1986), Singh (1988) and Giles (1991). Ohtani (1991) examines the properties of the ICLS estimator for a model with a single inequality constraint and multivariate t errors. He derives the first two moments of the ICLS estimator and numerically compares the bias and the mean square error (MSE) of the ICLS estimator with those of the unconstrained least squares estimator.

However, no similar analysis exists for the ICLS estimator with multiple inequality constraints and non-normal disturbances. In this paper, we generalize Ohtani’s (1991) analysis to a model with more than one inequality constraint. To keep our results tractable, and yet still allows for correlated parameter estimates, we follow the framework of Thomson (1982) by considering a regression model with two regressors, where each of the coefficients is subject to an inequality constraint. In Section 2, we outline the model and assumptions made, and derive the bias and MSE of the ICLS estimator under these assumptions. These results are numerically evaluated in Section 3 and comparisons are made with those for the case where the error terms are normally distributed. Section 4 presents conclusions.

2. THE MODEL AND THE MOMENTS OF THE ICLS ESTIMATOR

Consider the following model with two regressors:

\[ y = X\beta + Z\delta + \varepsilon, \quad \varepsilon \sim \text{Mt}(O, \sigma^2_n) \]  

(1)
where \( y \) is a \( n \times 1 \) vector of observations on the dependent variable; \( X \) and \( Z \) are \( n \times 1 \) vectors of observations on two explanatory variables; \( \beta \) and \( \delta \) are unknown coefficients. \( \varepsilon \), the vector of disturbances, is assumed to follow a multivariate \( t \) (Mt) distribution with probability density function given by

\[
f(\varepsilon | v, \sigma^2) = \frac{\Gamma\left(\frac{v+n}{2}\right)}{\sqrt{\pi^2 \Gamma(\frac{v}{2}) \sigma^n}} \left[1 + \frac{\varepsilon^\prime \varepsilon}{\sigma^2}\right]^{-(v+n)/2},
\]

where \( v \) and \( \sigma^2 \) are the degrees of freedom and scale parameters, and \( \sigma_c^2 = v \sigma^2/(v-2) \) is the common variance of the \( \varepsilon_i \)'s, \( i = 1, \ldots, n \), for \( v > 2 \). The normal distribution results when \( v = \infty \).

It is well known (see, for example Zellner (1976)) that the Mt distribution can be written as a combination of inverted Gamma and multivariate normal distributions. That is,

\[
\int_0^\infty f_N(\varepsilon | \theta) f_{IG}(\theta | v, \sigma^2) \, d\theta ,
\]

where

\[
f_N(\varepsilon | \theta) = \left(\frac{2\pi^{\frac{n}{2}} \theta^n}{\Gamma(n)}\right)^{-1} \exp\left(-\varepsilon^\prime \varepsilon / (2 \theta^2)\right)
\]

is the probability density function of \( \varepsilon \) when \( \varepsilon \sim N(0, \theta^2 I_n) \), and

\[
f_{IG}(\theta | v, \sigma^2) = \frac{2^{(v+1)/2} \Gamma\left(\frac{v}{2}\right) \sigma_v^{2/(v+1)}}{\Gamma\left(\frac{v}{2}\right) \sigma_v^{2/(v+1)}} \exp\left(-v \sigma^2/2 \theta^2\right)
\]

is the inverted Gamma density functions with degrees of freedom \( v \) and scale parameter \( \sigma^2 \).

In addition to sample information, it is assumed that each of the regression coefficients is subject to a non-negativity constraint. That is,

\[
\beta \geq 0 \quad \text{and} \quad \delta \geq 0 .
\]

(2)

The problem of estimating \( \beta \) and \( \delta \) subject to the inequality constraints \( \beta \geq 0 \) and \( \delta \geq 0 \) by the method of least squares can be handled as a quadratic programming problem. Forming the Lagrangian and solving the resultant Kuhn–Tucker conditions yield the following ICLS estimator.
\[
\begin{bmatrix}
\hat{\beta} \\
\hat{\delta}
\end{bmatrix} = \begin{cases}
(\bar{\beta}, \bar{\delta})' & \text{if } \bar{\beta} \geq 0 \text{ and } \bar{\delta} \geq 0 \\
(\bar{\beta}, 0)' & \text{if } \bar{\beta} \geq 0 \text{ and } \bar{\delta} < 0 \\
(0, \bar{\delta})' & \text{if } \bar{\beta} < 0 \text{ and } \bar{\delta} \geq 0 \\
(0, 0)' & \text{if } \bar{\beta} < 0 \text{ and } \bar{\delta} < 0
\end{cases}
\] (3)

where \(\bar{\beta} = \left[\Sigma Z^2 \Sigma XY - \Sigma XZ \Sigma ZY \right] / \left[\Sigma X^2 \Sigma Z^2 - (\Sigma XZ)^2 \right]\) and \(\bar{\delta} = \left[\Sigma X^2 \Sigma ZY - \Sigma XZ \Sigma Y \right] / \left[\Sigma X^2 \Sigma Z^2 - (\Sigma XZ)^2 \right]\) are the ordinary least squares (OLS) estimators of \(\beta\) and \(\delta\) respectively; and \(\bar{\beta} = \Sigma XY / \Sigma Z^2\) and \(\bar{\delta} = \Sigma ZY / \Sigma Z^2\) are the corresponding equality constrained least squares (ECLS) estimators.

Due to symmetry, we shall only consider the properties of \(\hat{\beta}\). Now, making use of the indicator functions, \(\hat{\beta}\) in (3) can be expressed as

\[
\hat{\beta} = \left[ I_{(0,0)}(\bar{\beta}) \right] [I_{(0,0)}(\bar{\delta})] \bar{\beta} + \left[ I_{(0,0)}(\bar{\beta}) \right] [I_{(-\infty,0)}(\bar{\delta})] \bar{\delta} ,
\]

where \(I_{(i)}(u)\) takes the value of 1 if \(u\) falls in the subscripted interval and 0 otherwise.

We let \(b = (\bar{\beta} - \beta)/\sigma_1\), \(d = (\bar{\delta} - \delta)/\sigma_2\), \(\bar{b} = (\bar{\beta} - \beta - d\Sigma XZ / \Sigma Z^2) / \sigma_3\), \(\bar{d} = (\bar{\delta} - \delta - b\Sigma XZ / \Sigma Z^2) / \sigma_4\), \(\tau = \beta / \sigma_1\), \(\eta = \delta / \sigma_2\) and \(\xi = (\beta + \delta \Sigma XZ / \Sigma Z^2) / \sigma_3\), where \(\sigma_i\)'s, \(i = 1, \ldots, 4\), are the standard errors of \(\bar{\beta}\), \(\bar{\delta}\), \(\bar{\beta}\) and \(\bar{\delta}\) respectively. Making use of these transformations, we can rewrite (4) as

\[
\hat{\beta} = \left[ I_{(-\infty,\omega)}(b) \right] [I_{(-\eta,\omega)}(d)] \left( \sigma_1 b + \beta \right) + \left[ I_{(-\infty,\omega)}(b) \right] [I_{(-\omega, -\eta)}(d)] \\
\times \left( \sigma_3 \bar{b} + \beta + \delta \Sigma XZ / \Sigma Z^2 \right)
\]

Furthermore, if we let \(\rho = -\left( \Sigma XZ / \Sigma Z^2 \right)^2 / \sigma_3 \) be the correlation coefficient between \(\bar{\beta}\) and \(\bar{\delta}\), recognizing that \(\sigma_3 / \sigma_1 = (1 - \rho^2)^{1/2}\) and after some manipulations, we can write \((\hat{\beta} - \beta) / \sigma_1\) as

\[
\left( \hat{\beta} - \beta \right) / \sigma_1 = \left[ I_{(-\infty,\omega)}(b) \right] [I_{(-\eta,\omega)}(d)] b - \left[ I_{(-\omega, -\tau)}(b) \right] \tau \\
- \left[ I_{(-\omega, -\eta)}(d) \right] \tau + \left[ I_{(-\omega, -\tau)}(b) \right] [I_{(-\omega,-\eta)}(d)] \tau
\]
\[ + \left[ I_{\xi,\omega}(b) \right] \left[ I_{\omega,\eta}(d) \right] \left( b(1 - \rho^2)^{1/2} \right) \left( \rho - \rho \eta \right) \]  

(5)

In order to evaluate the expectation of (5), we need the following supporting lemma.

**Lemma 1**: If \( b, d, \tau \) and \( \eta \) are defined as above and \( k \) is a constant, then

\[
E \left[ I_{1-\tau,\omega}(b) I_{1-\eta,\omega}(d)b^k \right] = \int_0^1 \int_0^1 \int_0^1 w_1^k \left[ 1 + \left( \frac{w_1^2}{2} - 2\rho w_1 w_2 + w_2^2 \right) \right] \left( \frac{\gamma(1 - \rho^2)}{2\pi(1 - \rho^2)^{1/2}} \right)^{-(v+2)/2} \left( 2\pi(1 - \rho^2)^{1/2} \right) \int_1^{t_1} \int_0^{t_2} dt_1 dt_2, \]

(6)

where \( t_1 = 1/(b + \tau + 1) \); \( t_2 = 1/(d + \eta + 1) \); \( w_1 = (1 - t_1 - \tau t_1)/t_1 \) and \( w_2 = (1 - t_2 - \eta t_2)/t_2 \).

**Proof**:

\[
E \left[ I_{1-\tau,\omega}(b) I_{1-\eta,\omega}(d)b^k \right] = \int_0^1 \int_0^1 \int_0^1 b^k \left[ \int_0^\infty f_N(b,d|\theta)f_\gamma(\theta|v, \sigma^2=1) \right] \, d\theta \, db \, dd
\]

\[
= C \int_0^1 \int_0^1 \int_0^1 b^k \theta^{-(v+3)} \exp \left( -\Omega/(2\theta^2) \right) \, d\theta \, dd \, db,
\]

(7)

where \( C = (v/2)^{v/2}/\left( \pi \Gamma(\nu/2)(1-\rho)^{1/2} \right) \) and \( \Omega = v + (b^2 - 2\rho bd + d^2)/(1 - \rho)^2 \).

Making use of the change in variable \( W = \Omega/\theta^2 \), and after some manipulations, we can write (7) as

\[
K \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 b^k \theta^{-(v+2)/2} \left[ W^{(v+2)/2-1} \exp \left( -W/2 \right) \right]/ \left( 2^{(v+2)/2} \Gamma(\nu/2) \right) \, dW \, dd \, db,
\]

(8)
where $K = (v/2)^{(v+2)/2} \frac{(v+2)/2-1}{\pi(1-\rho^2)^{1/2}}$.

After some further simplifications, (8) reduces to

$$E\left[(1-\tau, \omega)(1-\eta, \omega)(d)\beta^k\right] = \left[1/\left(2\pi(1-\rho^2)^{1/2}\right)\right] \int_{-\tau}^{\tau} \int_{-\eta}^{\eta} b^k \left[1 + (b - 2\rho bd + d)^2/v(1-\rho^2)^{-(v+2)/2}\right] dd \ db \quad (9)$$

For computational convenience, the ranges of integrations in (9) can be transformed to $[0, 1]$ by making use of the change in variables $t_1 = 1/(b + \tau + 1)$ and $t_2 = 1/(d + \eta + 1)$. Using this transformation and writing $(1 - t_1 - \tau t_1)/t_1$ and $(1 - t_2 - \eta t_2)/t_2$ as $w_1$ and $w_2$ respectively yield (6) directly.

Applying this Lemma, and using some results from Ohtani (1991), we can write the expected value of (5) as

$$E\left[\beta - \beta\right]/\sigma_1 = \int_0^1 \int_0^1 w_1 \left[1 + \left(w_1^2 - 2\rho w_1 w_2 + w_2^2\right)/v(1-\rho^2)^{-2}\right] \left[2\pi(1-\rho^2)^{1/2} w_1^2 t_1^2 t_2^2\right] dt_1 dt_2 - F(-t)\tau - F(-\eta)\tau + \phi(-\tau, -\eta, \rho)\tau + F(-\eta) \left[1 - (1-\rho^2)^{1/2} \Gamma\left(\frac{v+1}{2}\right) \right]$$

$$\times \int_0^1 \int_0^1 \left[1 + w_3^2/v\right]^{-2} \left[\eta \Gamma\left(\frac{v+1}{2}\right) w_3 t_3^2\right] dt_3 + (\tau - \rho\eta)(1 - F(-\xi)) \right], \quad (10)$$

where $t_3 = 1/(b + \xi + 1)$; $w_3 = (1 - t_3 - \xi t_3)/t_3$; $F(.)$ is the univariate $t$ cumulative distribution function and $\phi(.)$ is the probability integral of the bivariate $t$ distribution with correlation coefficient equals to $\rho$.

Similarly, the square of (5) can be expressed as

$$\left(\beta - \beta\right)^2/\sigma_1^2 = \left[I_{(-\tau, \omega)}(b)\right] \left[I_{(-\eta, \omega)}(d)\right] b^2 + \left[I_{(-\omega, -\tau)}(b)\right] \tau^2$$
\[ + \left[ I_{(\omega, \eta)}(d) \right] \tau^2 - \left[ I_{(\omega, \tau)}(b) \right] \left[ I_{(\omega, \eta)}(d) \right] \rho^2 \]
\[ + \left[ \Gamma_{(\xi, \omega)}(b) \right] \left[ I_{(\omega, \eta)}(d) \right] \left( 1 - \rho^2 \right) \]
\[ - 2\sqrt{1 - \rho^2} \rho_\eta + \rho^2 \eta^2 - \tau^2 \]

(11)

Applying Lemma 1 and the results of Ohtani (1991) again, the MSE of \( \hat{\beta} \) relative to \( \sigma_1^2 \) can be expressed as

\[
\text{MSE}(\hat{\beta})/\sigma_1^2 = \text{E}\left( \hat{\beta} - \beta \right)^2 / \sigma_1^2
\]
\[
= \left[ \int \int \left( w_1^2 \left[ 1 + \left( w_1^2 - 2 \rho w_1 w_2 + w_2^2 \right) / \left( v(1 - \rho^2) \right) \right] \right)^{(v+2)/2} \]
\[ \times \left\{ 2\pi(1 - \rho^2)^{1/2} \tau_1^2 \right\} dt_1 dt_2 + F(-\tau) \tau^2 + F(-\eta) \tau^2 \]
\[ - \phi(-\tau, -\eta, \rho) \tau^2 + F(-\eta) \left[ \Gamma\left( \frac{v+1}{2} \right) \right] \int_0^1 w_3 (1 - \rho^2)^{1/2} \]
\[ \times \left( w_3 (1 - \rho^2)^{1/2 - 2 \rho \eta} \right) \left( 1 + w_3^2 / v \right)^{-(v+1)/2} \]
\[ \times \left( v \pi \right)^{1/2} \Gamma\left( \frac{v}{2} \right) \right] \int_0^1 \left( \rho^2 \eta^2 - \tau^2 \right)(1 - F(-\xi)) \]

(12)

3. NUMERICAL COMPUTATION OF RESULTS

As it appears impossible to further analyze the relative bias and MSE of \( \hat{\beta} \), we carried out numerical calculations of (10) and (12) for \( v = 5, 10, 25; \eta = 20, -5, -2, 0, 2, 5, 20; \beta = [-5, 5] \) and \( \rho = -0.7, -0.5, -0.4, 0, 0.4, 0.5, 0.7 \). Given these parameter values, \( \xi \) can be calculated readily by noting that \( \xi = (\tau - \rho \eta)/(1 - \rho^2)^{1/2} \). The Gamma and the univariate \( t \) cumulative distribution functions were computed using the subroutines GAMMLN and BETAI from Press et al. (1986). The expected values of the univariate and bivariate \( t \) variable were calculated using various subroutines from the NAG (1991) library. For purposes of comparison, the corresponding expressions given in Thomson
(1982) for the multivariate normal error case were also evaluated. Some representative results appear in Figures 1 to 3 and Table 1.

From the figures and table, we observe the following. Firstly, regardless of the magnitude of \( \eta \) and \( v \), when \( \rho = 0 \) (i.e., the regressors are orthogonal), the validity of the second constraint has
Figure 3: Bias of the ICLS estimator ($\eta = 2$)

no bearing on the properties of $\hat{\beta}$, and consequently the bias and MSE of $\hat{\beta}$ behave as in the single inequality constraint case (Ohtani (1991)). Here, as $\beta \to -\infty$, both the bias and MSE of $\hat{\beta}$ increases without limit; as $\beta \to +\infty$, the MSE and bias of $\hat{\beta}$ approach the corresponding MSE and bias of $\bar{\beta}$, the unrestricted least squares estimator. Accordingly, $\hat{\beta}$ is unbiased only if the true value of $\beta$ is sufficiently large. When $\rho = 0$ and the constraint is true or nearly so, $\hat{\beta}$ has a smaller MSE than $\bar{\beta}$. Also, a decrease of $\nu$ from infinity (the normal error case) causes both the bias and MSE of $\hat{\beta}$ to shift upwards, and decreases the respective rate at which the bias of $\hat{\beta}$ approaches zero and the MSE of $\hat{\beta}$ approaches that of the unrestricted estimator. Regardless of the value of $\nu$, when $\rho = 0$, the maximum gain in MSE of the ICLS estimator over the OLS estimator occurs at $\beta = 0$.

As in the case of normal distributed error terms (Thomson (1982)), these observations do not necessarily carry over to the case in which the regressors are non-orthogonal. For any given $\eta$, $\nu$ and $\tau$, the bias of $\hat{\beta}$ increases (decreases) as $\rho$ increases (decreases). When $\eta$ is negative and $|\eta|$ is large (Figure 1), the ICLS estimator with $\rho > 0$ is positively biased for all values of $\beta$. In contrast, the bias of $\hat{\beta}$ with
<table>
<thead>
<tr>
<th>$\eta = \delta/\sigma$</th>
<th>$\tau = \beta/\sigma_1$</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>OLS($v=\infty$)</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>OLS($v=5$)</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=\infty, p=0$)</td>
<td>16.00</td>
<td>9.00</td>
<td>4.04</td>
<td>1.24</td>
<td>0.50</td>
<td>0.76</td>
<td>0.96</td>
<td>0.99</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=5, p=0$)</td>
<td>16.05</td>
<td>9.11</td>
<td>4.24</td>
<td>1.51</td>
<td>0.78</td>
<td>1.05</td>
<td>1.33</td>
<td>1.46</td>
<td>1.52</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=\infty, p=0.5$)</td>
<td>16.07</td>
<td>9.06</td>
<td>4.27</td>
<td>2.08</td>
<td>1.72</td>
<td>1.75</td>
<td>1.76</td>
<td>1.76</td>
<td>1.77</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=5, p=0.5$)</td>
<td>16.14</td>
<td>9.29</td>
<td>4.70</td>
<td>2.58</td>
<td>2.22</td>
<td>2.28</td>
<td>2.33</td>
<td>2.36</td>
<td>2.37</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=\infty, p=-0.5$)</td>
<td>16.07</td>
<td>9.04</td>
<td>4.02</td>
<td>1.01</td>
<td>0.04</td>
<td>0.68</td>
<td>1.50</td>
<td>1.74</td>
<td>1.76</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=5, p=-0.5$)</td>
<td>16.02</td>
<td>9.04</td>
<td>4.06</td>
<td>1.11</td>
<td>0.18</td>
<td>0.82</td>
<td>1.70</td>
<td>2.10</td>
<td>2.26</td>
</tr>
<tr>
<td>0</td>
<td>OLS($v=\infty$)</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>OLS($v=5$)</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=\infty, p=0$)</td>
<td>16.00</td>
<td>9.00</td>
<td>4.04</td>
<td>1.24</td>
<td>0.50</td>
<td>0.76</td>
<td>0.96</td>
<td>0.99</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=5, p=0$)</td>
<td>16.05</td>
<td>9.11</td>
<td>4.24</td>
<td>1.51</td>
<td>0.78</td>
<td>1.05</td>
<td>1.33</td>
<td>1.46</td>
<td>1.51</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=\infty, p=0.4$)</td>
<td>16.00</td>
<td>9.00</td>
<td>4.05</td>
<td>1.28</td>
<td>0.58</td>
<td>0.78</td>
<td>0.90</td>
<td>0.92</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=5, p=0.4$)</td>
<td>16.08</td>
<td>9.16</td>
<td>4.32</td>
<td>1.66</td>
<td>0.97</td>
<td>1.19</td>
<td>1.38</td>
<td>1.46</td>
<td>1.50</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=\infty, p=-0.4$)</td>
<td>16.00</td>
<td>9.00</td>
<td>4.02</td>
<td>1.14</td>
<td>0.33</td>
<td>0.64</td>
<td>0.87</td>
<td>0.92</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=5, p=-0.4$)</td>
<td>16.04</td>
<td>9.07</td>
<td>4.15</td>
<td>1.35</td>
<td>0.56</td>
<td>0.88</td>
<td>1.22</td>
<td>1.38</td>
<td>1.45</td>
</tr>
<tr>
<td>2</td>
<td>OLS($v=\infty$)</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>OLS($v=5$)</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
<td>1.67</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=\infty, p=0$)</td>
<td>16.00</td>
<td>9.00</td>
<td>4.04</td>
<td>1.24</td>
<td>0.50</td>
<td>0.76</td>
<td>0.96</td>
<td>0.99</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=5, p=0$)</td>
<td>16.05</td>
<td>9.11</td>
<td>4.24</td>
<td>1.51</td>
<td>0.78</td>
<td>1.05</td>
<td>1.33</td>
<td>1.46</td>
<td>1.51</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=\infty, p=0.7$)</td>
<td>15.92</td>
<td>9.01</td>
<td>4.05</td>
<td>1.24</td>
<td>0.50</td>
<td>0.76</td>
<td>0.95</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=5, p=0.7$)</td>
<td>16.06</td>
<td>9.13</td>
<td>4.27</td>
<td>1.56</td>
<td>0.83</td>
<td>1.10</td>
<td>1.36</td>
<td>1.45</td>
<td>1.47</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=\infty, p=-0.7$)</td>
<td>16.00</td>
<td>9.00</td>
<td>4.03</td>
<td>1.22</td>
<td>0.48</td>
<td>0.74</td>
<td>0.93</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>ICLS($v=5, p=-0.7$)</td>
<td>16.01</td>
<td>9.03</td>
<td>4.12</td>
<td>1.38</td>
<td>0.65</td>
<td>0.92</td>
<td>1.21</td>
<td>1.35</td>
<td>1.41</td>
</tr>
</tbody>
</table>
\( \rho < 0 \) is positive only in the region where \( \beta \) is negative or close to zero, and is negative when \( \beta \) is positive. As \( \eta \) increases from being negative towards being positive (Figure 3), the biases of the ICLS estimators with \( \rho \neq 0 \) behave as in the orthogonal regressors case. For any given \( \eta \) and \( \tau \), a departure of the error terms from normality shifts the bias of the ICLS estimator upwards. However, at least for the cases that we have considered, the effects of decreasing \( \nu \) from the normal case on the bias of \( \hat{\beta} \) is typically negligible.

In terms of the mean square error comparison, our numerical results show that for any given values of \( \eta \) and \( \nu \), a negative correlation coefficient increases the maximum gain of using the ICLS over OLS estimator, and the potential gain is maximized as \( \eta \to -\infty \). On the other hand, if \( \rho > 0 \), the minimum of MSE(\( \beta \)) is no less than half that of MSE(\( \beta \)) and the potential gain of using \( \hat{\beta} \) over \( \tilde{\beta} \) is maximized as \( \eta \to \infty \). At least for the cases that we have considered, when \( \rho > 0 \) and the true value of \( \eta \) is negative and \( |\eta| \) is large, the OLS estimator strictly dominates the ICLS estimator. Moreover, regardless of the value of \( \nu \), when \( \eta < 0 \) and \( \rho < 0 \), the ICLS estimator is superior to the OLS estimator only when \( \beta \) is close to zero, and can have a larger MSE than that of the OLS estimator in the region where \( \beta > 0 \). These results suggest that imposing a correctly specified restriction does not necessarily guarantee an improvement in estimation efficiency when the regressors are correlated and the validity of other constraints in the model is uncertain. These are consistent with the findings of Thomson (1982) when the regression disturbances are assumed to be normally distributed. Furthermore, similar to the behaviour of the bias, when \( \eta \to \infty \), the MSE of \( \hat{\beta} \) behaves as in the one constraint case. Intuitively, if \( \eta \) is large, the second constraint becomes ineffective, hence the bias and MSE of \( \hat{\beta} \) behave as if only \( \beta \) is constrained. This feature is observed for a relatively small (non-negative) \( \eta \) when \( \nu \) is large, but only for a sufficiently large \( \eta \) when \( \nu \) is small. Table 1 reports some of these results.

4. CONCLUSION

We have considered the sampling properties of the inequality
constrained least squares estimator in a model with two non-negativity restrictions and multivariate t error terms. Numerical evaluations of the bias and mean square error expressions suggest that the properties of the estimator with Mt errors are qualitatively consistent with those found for the model with normally distributed error terms. In particular, the statistical properties of the inequality constrained least squares estimator are dependent on the correlation between the constrained estimates. An incorrect constraint on one coefficient can lead to an increase in mean square error of estimators of other coefficients on which correctly specified inequality restrictions are imposed. Although a decrease in the value of the degrees of freedom parameter from the case of normally distributed error terms has quantitative impacts on the bias and mean square error of the ICLS estimator, the effects are relatively insignificant, at least for the cases that we have considered. This suggests that the properties of the inequality constrained least squares estimator under normal errors are robust to this alternative specification of the error distribution.

ACKNOWLEDGEMENTS

I am grateful to Robert Bartels, Trevor Breusch, Murray Smith and Arnold Zellner for their helpful comments.

BIBLIOGRAPHY


Ohtani, K. (1991), "Small sample properties of the interval constrained least squares estimator when error terms have a multivariate t distribution", Journal of the Japan Statistical Society 21, 197–204.


Received April, 1994; Revised June, 1995.