THE NON-OPTIMALITY OF INTERVAL RESTRICTED
AND PRE-TEST ESTIMATORS UNDER
SQUARED ERROR LOSS

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ABSTRACT

We consider the linear regression model with an interval
restriction imposed on the coefficients, and examine the sampling
performance of a family of Stein interval restricted and pre-test
estimators for the coefficient vector. The risk, under squared error
loss, of these Stein-like estimators are derived, and the
inadmissibility of the maximum likelihood interval restricted and
pre-test estimators is demonstrated.

1. INTRODUCTION

When estimating the parameters of a linear regression model, prior
information often requires certain regression coefficients to lie
within fixed intervals. This problem is usually handled by maximizing
the likelihood function subject to a system of inequalities. There is
a considerable body of literature relating to the sampling properties
of the resulting interval restricted maximum likelihood (INML)
estimator. Examples are Escobar and Skarpness (1986, 1987) and Ohtani
performance of an interval pre-test (INPT) estimator, when estimation
is conditional upon a preliminary test for an interval constraint on a
linear combination of coefficients.

It is well known that under a measure such as squared error loss,
the traditional maximum likelihood estimator for the k-dimensional
coefficient vector is dominated by the Stein-rule estimator when k ≥ 3
and is therefore inadmissible (See, for example, James and Stein
(1961), or Brown (1966)). Furthermore, in the context of a linear
regression model with equality restrictions on the coefficients, Sclove
et al. (1972) proposed a modified pre-test estimator which combines the
positive-part version of the Stein-rule estimator and the linear
restrictions according to a pre-test rule. They show that this
modified pre-test estimator uniformly dominates the traditional
equality pre-test estimator. However, the efficiency of various
Stein-like estimators, when prior information in the form of an
interval restriction is introduced along with sample information, has
yet to be explored. Judge et al. (1984) and Judge and Yancey (1986)
have considered a Stein inequality restricted estimator, but their
constraint is not of an interval form.

The purpose of this paper is to extend the known results reported
in the literature in two directions. First, we generalise the results
of Judge et al. (1984) and Judge and Yancey (1986) to include an
interval constraint. Second, we define a family of Stein interval
pre-test estimators, which nest the corresponding estimators with an
inequality constraint as a special case, and explore their sampling
performance using the risk under squared error loss measure. We show
that the INML and INPT estimators are both inadmissible, and in
particular, the traditional Stein-rule estimator and its positive-part
variant can strictly dominate their corresponding interval restricted
and pre-test estimators.
2. MODEL FRAMEWORK AND THE MAXIMUM LIKELIHOOD ESTIMATORS

Consider the linear model
\[ y = X\beta + \varepsilon ; \quad \varepsilon \sim N(0, \sigma^2 I) \quad \text{(1)} \]
where \( y \) and \( \varepsilon \) are \( n \times 1 \), \( \beta \) is \( k \times 1 \) and \( X \) is \( n \times k \), non-stochastic and of rank \( k \). In addition to sample information, we assume that there exists prior information that a linear combination of the coefficients belongs to an interval constraint of the form
\[ r_1 \leq C'\beta \leq r_2 , \quad \text{(2)} \]
where \( r_1 \) and \( r_2 \) are known scalars \((r_1 < r_2)\), and \( C' \) is a \( 1 \times k \) known vector.

Following Judge and Yancey (1981, 1986), we can reparameterize (1) and (2) as
\[ y = H\theta + \varepsilon \quad \text{(3)} \]
and
\[ \rho_1 \leq \theta_1 \leq \rho_2 \quad \text{(4)} \]
respectively, where \( H = XS^{-1/2}Q' \); \( S = X'X \); \( \theta = QS^{1/2}\beta \); \( \theta_1 \) is the first element of \( \theta \); \( \rho_1 = r_1/h_1 \); \( \rho_2 = r_2/h_1 \); \( h_1 \) is the first element of \( h' = C'S^{-1/2}Q' \) and is assumed to be positive without loss of generality; and \( Q \) is an orthogonal matrix such that \( QS^{-1/2}C(S^{-1/2}C)'C'S^{-1/2}Q' = \begin{pmatrix} 1 & 0' \\ 0 & 0 \end{pmatrix} \).

The INML estimator of \( \theta \) is given by
\[ \theta^* = \begin{cases} \theta_L^* & \text{if } \tilde{\theta}_1 < \rho_1 \\ \tilde{\theta} & \text{if } \rho_1 \leq \tilde{\theta}_1 \leq \rho_2 \\ \theta_R^* & \text{if } \tilde{\theta}_1 > \rho_2 \end{cases} \quad \text{(5)} \]

\[ = I_{(-\infty, \rho_1)}(\tilde{\theta})\theta_L^* + I_{[\rho_1, \rho_2]}(\tilde{\theta})\tilde{\theta} + I_{(\rho_2, \infty)}(\tilde{\theta})\theta_R^* \]
where \( \tilde{\theta} = H'y \) is the unrestricted maximum likelihood estimator of \( \theta \); \( \tilde{\theta}_1 \) is the first element of \( \tilde{\theta} \); \( \theta_L^* = \left( \begin{array}{c} \rho_1 \\ \tilde{\theta}_{(k-1)} \end{array} \right) \) and \( \theta_R^* = \left( \begin{array}{c} \rho_2 \\ \tilde{\theta}_{(k-1)} \end{array} \right) \) are the equality restricted maximum likelihood estimators corresponding to imposing \( \theta_1 = \rho_1 \) and \( \theta_1 = \rho_2 \) respectively; \( \tilde{\theta}_{(k-1)} = \left( 0, \mathbb{I}_{(k-1)} \right) \tilde{\theta} \); and \( I_{(.)}(u) \) is an indicator function which takes the value of 1 if \( u \) falls
in the subscripted interval and 0 otherwise.

Now, the relative risk of $\theta^{**}$, under squared error loss, is defined as

$$
\rho(\theta^{**}, \theta) = E\left[(\theta^{**} - \theta)'(\theta^{**} - \theta)\right]/\sigma^2 \quad (6)
$$

Using a simple generalisation of the result given in Ohtani (1987), (6) can be written as

$$
\rho(\theta^{**}, \theta) = k - 1 + (\delta_1/\sigma)F(\delta_1/\sigma) + (\delta_2/\sigma)f(\delta_2/\sigma)
- (\delta_1/\sigma)f(\delta_2/\sigma) + F(\delta_2/\sigma) - F(\delta_1/\sigma)
+ (\delta_2/\sigma)^2[1 - F(\delta_2/\sigma)] \quad (7)
$$

where $\delta_i = \rho_i - \theta$, $i = 1, 2$; and $f(.)$ and $F(.)$ are respectively the probability density function and the cumulative distribution function of the standard normal distribution.

The researcher is typically uncertain of the validity of the interval constraint and may test for the interval restriction (4) based on a two stage test. In the first stage, he tests the following:

$$
H_{10} : \theta_1 \geq \rho_1 \quad \text{vs.} \quad H_{11} : \theta_1 < \rho_1 \quad (8)
$$

Assuming that $\sigma^2$ is known, the test statistic for (8) is $u_1 = (\tilde{\theta}_1 - \rho_1)/\sigma$, which has a standard normal distribution when $\theta_1 = \rho_1$. Given the test statistic, the researcher may reject $H_{10}$ if $u_1 < c_1 \leq 0$ and uses the unrestricted estimator $\tilde{\theta}$, where $c_1$ is the size $- \alpha_1$ critical value for the standard normal random variable. Alternatively, he may accept $H_{10}$ if $u_1 \geq c_1$ and proceed to the second stage in which he tests the following hypothesis:

$$
H_{20} : \theta_1 \leq \rho_2 \quad \text{vs.} \quad H_{21} : \theta_1 > \rho_2 \quad (9)
$$

The test statistic for (9) is $u_2 = (\tilde{\theta}_1 - \rho_2)/\sigma$. As the second stage test depends on the result of the first stage, the critical value $c_2 \geq 0$ corresponding to the size $\alpha_2$ of the test for (9) must satisfy the following:

$$
\alpha_2 = P\left[ u_2 \geq c_2 | (u_1 \geq c_1 \text{ and } H_{20} \text{ is true}) \right]. \quad (10)
$$

For simplicity, we assume that $\alpha_1 = \alpha_2 = \alpha$. Under this assumption, Hasegawa (1991) shows that $c_2$ can be determined by considering the relationship
\[ P(u > c_2) = \alpha F(-c_1 + d), \]  

where \( u = (\tilde{\theta} - \theta)/\sigma \) and \( d = (\rho_2 - \rho_1)/\sigma \). The values of \( c_1 \) and \( c_2 \) for \( d = 0.5, 1.0, 1.5, \ldots, 5.0 \) and \( \alpha = 0.025, 0.05 \) are given in Hasegawa (1991).

The researcher may reject \( H_{20} \) if \( u_2 \geq c_2 \) and uses \( \tilde{\theta} \). Alternatively, he may accept \( H_{20} \) if \( u_2 < c_2 \) and uses the INML estimator \( \theta^{**} \). This two stage test gives rise to the following interval pre-test estimator

\[
\hat{\theta} = I_{\{c \leq \omega, c \leq \tilde{c}\}}(u) + I_{\{c \geq \omega, c \geq \tilde{c}\}}(u)\tilde{\theta} + I_{\{c < \omega, c < \tilde{c}\}}(u)\theta^{**}
\]

\[
= \tilde{\theta} - \left[ I_{\{c < \omega, c < \tilde{c}\}}(u) - I_{\{c < \omega, c < \tilde{c}\}}(u) \right] \times (\tilde{\theta} - \theta^{**})
\]

(12)

Using a simple generalisation of the results given in Hasegawa (1991), it is possible to write the risk of \( \hat{\theta} \) as

\[
\rho(\hat{\theta}, \theta) = k + (\delta/\sigma)I(\delta/\sigma) - (\delta/\sigma) \int \frac{d \sigma}{\sigma}
\]

\[
+ F(\delta/\sigma) - F(\delta/\sigma) - (c + \delta/\sigma)I(\delta/\sigma)
\]

\[
+ (c + \delta/\sigma)I(\delta/\sigma) - F(c + \delta/\sigma) + F(c + \delta/\sigma)
\]

\[
+ (\delta/\sigma)^2 [F(c + \delta/\sigma) - F(\delta/\sigma)]
\]

\[
+ (\delta/\sigma)^2 [F(\delta/\sigma) - F(c + \delta/\sigma)]
\]

(13)

Numerical evaluations of these risk functions show that both \( \hat{\theta} \) and \( \theta^{**} \) are risk superior to \( \tilde{\theta} \) when the interval restriction is true or close enough to being true (see Ohtani (1987) and Hasegawa (1991)). In addition, the shape of these risk functions depends largely on \( d \), the length of the interval constraint relative to \( \sigma \), rather than on \( \rho_1/\sigma \) and \( \rho_2/\sigma \) themselves. Also, as one of the endpoints of the interval approaches positive or negative infinity, \( \theta^{**} \) and \( \hat{\theta} \) collapse respectively to the inequality restricted maximum likelihood (IRML) and inequality pre-test (IPT) estimators discussed in the literature (see Judge and Yancey (1986) for a list of references).

In the next section we introduce an alternative family of estimators. Furthermore, we show that both the INML and INPT estimators are dominated by the corresponding interval restricted and pre-test estimators within this family.
3. THE INTERVAL STEIN RESTRICTED AND PRE-TEST ESTIMATORS

Within the context of model (3), and assuming that $\sigma^2$ is known, James and Stein (1961) show that when $k \geq 3$, the unrestricted maximum likelihood estimator $\tilde{\theta}$ is uniformly dominated by the Stein-rule estimator

$$
\tilde{\theta}_s = (1 - a\sigma^2/\tilde{\theta}')\tilde{\theta}
$$

(14)

where $0 \leq a \leq 2(k-2)$. The risk of $\tilde{\theta}_s$ is given by

$$
\rho(\tilde{\theta}_s, \theta) = k - a\left(2(k-2) - a\right)E\left[1/\chi^2_{1;\lambda}\right]
$$

(15)

where $\lambda = \theta'/\theta(2\sigma^2)$. $\tilde{\theta}_s$ is minimax, and the minimal risk of $\tilde{\theta}_s$ occurs when $a = k-2$.

Baranchik (1964) and Stein (1966) later show that $\tilde{\theta}_s$ is dominated by the positive-part Stein-rule estimator

$$
\tilde{\theta}_s^* = \mathbb{1}_{[a,\infty)}(\tilde{\theta}'\tilde{\theta}/\sigma^2)(1 - a\sigma^2/\tilde{\theta}'\tilde{\theta})\tilde{\theta}
$$

(16)

with risk equal to

$$
\rho(\tilde{\theta}_s^*, \theta) = \rho(\tilde{\theta}_s, \theta) - 4\lambda E\left[\mathbb{1}_{[0,\infty)}(\chi^2_{1;\lambda})(1-a/\chi^2_{(k-1);\lambda})\right]
$$

(17)

$$
- 4\lambda E\left[\mathbb{1}_{[0,\infty)}(\chi^2_{(k-2);\lambda})(a/\chi^2_{(k+2);\lambda})^{-1}\right]
$$

where as before $0 \leq a \leq 2(k-2)$. $\tilde{\theta}_s^*$ is also minimax. Unfortunately, there is no single value of $a$ that minimizes the risk of $\tilde{\theta}_s^*$.

Given these results, it seems intuitive that if the Stein-rule estimator $\tilde{\theta}_s$ or its positive-rule counterpart $\tilde{\theta}_s^*$ is used in place of the unrestricted estimator $\tilde{\theta}$, and when $k \geq 4$ the $\tilde{\theta}_s^{(k-1)}$ components of $\tilde{\theta}_L$ and $\tilde{\theta}_R$ in (5) and (12), then the resultant estimators might be superior to the interval restricted and pre-test estimators based on the maximum likelihood rule. This is similar to the approach used by Scolve et al. (1972) in specifying a modified pre-test estimator when the hypothesis of interest is in the form of linear equalities. Based on the same approach, Judge et al. (1984) propose a Stein inequality restricted estimator, which is a special case of (18).

Now, when $k \geq 4$, the Stein interval restricted and pre-test estimators of $\theta$ are defined as
\[
\theta_{**} = I_{(\omega, \rho_1)}(\tilde{\theta}) \left[ \rho_1 \right] (1 - a_1 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]
\[
+ I_{(\rho_1, \rho_2)}(\tilde{\theta}) \left[ \rho_2 \right] (1 - a_2 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]
\[
+ I_{(\rho_2, \omega)}(\tilde{\theta}) \left[ \rho_2 \right] (1 - a_3 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]
\[
= \theta_{**} - I_{(\omega, \rho_1)}(\tilde{\theta}) (a_1 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta}) - I_{(\rho_1, \rho_2)}(\tilde{\theta}) (a_2 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]
\[
- I_{(\rho_2, \omega)}(\tilde{\theta}) (a_3 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]

and
\[
\hat{\theta}_s = \hat{\theta} - \left[ I_{(\omega, c_2)}(u_2) - I_{(\omega, c_1)}(u_1) \right] (\tilde{\theta} - \theta_{**})
\]
\[
= \hat{\theta} - \left[ 1 - \left( I_{(\omega, c_2)}(u_2) - I_{(\omega, c_1)}(u_1) \right) (a_1 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]
\[
- \left[ I_{(\omega, c_2)}(u_2) - I_{(\omega, c_1)}(u_1) \right] \times \left[ I_{(\omega, \rho_1)}(\tilde{\theta}) \right]
\]
\[
\times (a_2 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta}) + I_{(\rho_1, \rho_2)}(\tilde{\theta}) (a_3 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]
\[
+ I_{(\rho_2, \omega)}(\tilde{\theta}) (a_3 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]

respectively, where \( \tilde{\theta}_0 = (0, \tilde{\theta} (k-1)) \). If \( \rho_1 \to -\omega \) or \( \rho_2 \to \omega \), then (18) collapses to the inequality Stein restricted estimator given in Judge et al. (1984).

Similarly, we can form the positive-part Stein interval restricted and pre-test estimators by replacing \( \hat{\theta}_s \) by \( \hat{\theta}_s^* \). It is readily shown that when \( k \geq 4 \), the positive-part Stein interval restricted and pre-test estimators can be written as
\[
\theta_{**} = \theta_{**} - I_{(\omega, \rho_1)}(\tilde{\theta}) \left[ 1 \right] (\tilde{\sigma} \tilde{\theta} / \sigma^2) (1 - a_1 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]
\[
+ I_{(\rho_1, \rho_2)}(\tilde{\theta}) \left[ 1 \right] (\tilde{\sigma} \tilde{\theta} / \sigma^2) (1 - a_2 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]
\[
- I_{(\rho_2, \omega)}(\tilde{\theta}) \left[ 1 \right] (\tilde{\sigma} \tilde{\theta} / \sigma^2) (1 - a_3 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]

and
\[
\hat{\theta}_s^* = \hat{\theta}_s - \left[ 1 - \left( I_{(\omega, c_2)}(u_2) - I_{(\omega, c_1)}(u_1) \right) \right] (\tilde{\sigma} \tilde{\theta} / \sigma^2)
\]
\[
+ \left[ I_{(\rho_1, \rho_2)}(\tilde{\theta}) \right] (1 - a_2 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta})
\]
\[
\times (1 - a_2 \sigma^2 / \tilde{\sigma} \tilde{\theta} + (k-1) \tilde{\theta} \tilde{\sigma} \tilde{\theta}) - \left[ I_{(\omega, c_2)}(u_2) - I_{(\omega, c_1)}(u_1) \right]
\]
\[
\begin{align*}
\times & \left\{ 1_{(-\omega_1, \rho_1)}(\tilde{\theta}_1) \left(I_{(0, \lambda)}(\tilde{\theta}_0/\sigma^2)(1 - a_1 \sigma^2/\tilde{\theta}_0 \tilde{\theta}_0) \tilde{\theta}_0 \right)
+ I_{(\rho_1, \rho_2)}(\tilde{\theta}_1) \left(I_{(0, \lambda)}(\tilde{\theta}_0/\sigma^2)(1 - a_2 \sigma^2/\tilde{\theta}_0 \tilde{\theta}_0) \tilde{\theta}_0 \right)
+ I_{(\rho_2, \omega)}(\tilde{\theta}_1) \left(I_{(0, \lambda)}(\tilde{\theta}_0/\sigma^2)(1 - a_3 \sigma^2/\tilde{\theta}_0 \tilde{\theta}_0) \tilde{\theta}_0 \right) \right\} \\
\end{align*}
\] 

(21)

respectively. An alternative way to formulate \( \hat{\theta}_s \) and \( \hat{\theta}_s^* \) is to use the entire \( \tilde{\theta} \) in the shrinkage factor of the first and third components in (18) and (19). Similar estimators can be defined from the positive part theory, and their risk functions can be studied. This remains an interesting point of departure for future research.

We now derive the expressions for the relative risk functions of \( \hat{\theta}_s^*, \hat{\theta}_s, \hat{\theta}_s**, \) and \( \hat{\theta}_s^* \).

**Theorem 1.** Under the stated assumptions,

\[
\rho(\theta_s^*, \theta) = \rho(\theta_s**, \theta) + F(\delta_1/\sigma) a_1 \left( a_1 - 2(k-3) \right) E \left[ 1/\chi^2_{(k-1; \lambda_0)} \right] \\
+ E \left[ I_{(\rho_1, \rho_2)}(\tilde{\theta}_1) a_2 \left( a_2 - 2(k-2) \right) \left( 1/\chi^2_{(k; \lambda)} \right) \right] \\
+ \left( 1 - F(\delta_2/\sigma) \right) a_3 \left( a_3 - 2(k-3) \right) E \left[ 1/\chi^2_{(k-1; \lambda_0)} \right].
\]

(22)

\[
\rho(\hat{\theta}_s^*, \theta) = \rho(\hat{\theta}_s, \theta) + \left( F(\delta_1/\sigma) - F(\delta_1/\sigma) \right) a_1 \left( a_1 - 2(k-3) \right) E \left[ 1/\chi^2_{(k-1; \lambda_0)} \right] \\
+ E \left[ I_{(\rho_1, \rho_2)}(\tilde{\theta}_1) a_2 \left( a_2 - 2(k-2) \right) \left( 1/\chi^2_{(k; \lambda)} \right) \right] \\
+ \left( F(\delta_2/\sigma) - F(\delta_2/\sigma) \right) \\
\times a_3 \left( a_3 - 2(k-3) \right) E \left[ 1/\chi^2_{(k-1; \lambda_0)} \right].
\]

(23)

\[
\rho(\theta_s**, \theta) = \rho(\theta_s**, \theta) + F(\delta_1/\sigma) \left( 2a_1 N(a_1, k-1, \lambda_0) + \left( 4 \lambda_0 - (k-1) \right) \right) \\
\times N(a_1, k+1, \lambda_0) - 2 \lambda_0 N(a_1, k+3, \lambda_0) - a^2 E \left[ I_{(0, a_1)} \right] \\
\times \left( \chi^2_{(k-1; \lambda_0)} / \chi^2_{(k-1; \lambda_0)} \right) - 4 \lambda_0 E \left[ I_{(0, a_1)} \chi^2_{(k-1; \lambda_0)} \right] \\
\times (\chi^2_{(k-1; \lambda_0)}/\chi^2_{(k-1; \lambda_0)}) - 4 \lambda_0 E \left[ I_{(0, a_1)} (\chi^2_{(k-1; \lambda_0)}) \right].
\]
\[ \begin{align*}
\chi^2_{(k+1; \lambda_0)} & \geq \left[ 1 - F(\delta_2 / \sigma) \right] \left( 2a_3 N(a_3, k-1, \lambda_0) + \frac{4\lambda_0 - (k-1)N(a_3, k+1, \lambda_0) - 2\lambda_0 N(a_3, k+3, \lambda_0) - a_3^2 E[I(0,a_3)]}{2a_2^2 / (\delta_2^2 + \chi^2_{(k-1; \lambda_0)})} \right) \\
\chi^2_{(k+1; \lambda_0)} & \geq 4a_1^2 \lambda_0 E[I(0,a_3)](\chi^2_{(k+1; \lambda_0)}) \\
\chi^2_{(k+1; \lambda_0)} & \geq \frac{\left( 2a_2 - a_2^2 / (\delta_2^2 + \chi^2_{(k-1; \lambda_0)}) \right) - \delta_1^2 / \sigma^2 + 2\theta_1 \tilde{\theta}_1 - 2a_2 \theta_1 \tilde{\theta}_1}{(\delta_2^2 + \chi^2_{(k-1; \lambda_0)}) + 2a_2 (k-1) / (\delta_2^2 + \chi^2_{(k-1; \lambda_0)})} \\
\chi^2_{(k+1; \lambda_0)} & \geq 4a_1^2 \lambda_0 - (k-1) - 2a_2 (k-1) \\
\chi^2_{(k+1; \lambda_0)} & \geq 8a_2^2 \lambda_0 / (\delta_2^2 + \chi^2_{(k-1; \lambda_0)})^2 \right]\end{align*} \]

and

\[ \rho(\hat{\theta}^*, \theta) = \rho(\hat{\theta}^*, \theta) + \left( F(\delta_1 / \sigma) - F(c_1 + \delta_1 / \sigma) \right) \left( 2a_1 N(a_1, k-1, \lambda_0) + \frac{4\lambda_0 - (k-1)N(a_1, k+1, \lambda_0) - 2\lambda_0 N(a_1, k+3, \lambda_0) - a_1^2 E[I(0,a_1)]}{2a_2^2 / (\delta_2^2 + \chi^2_{(k-1; \lambda_0)})} \right) \\
\chi^2_{(k+1; \lambda_0)} & \geq 4a_1^2 \lambda_0 E[I(0,a_3)](\chi^2_{(k+1; \lambda_0)}) \\
\chi^2_{(k+1; \lambda_0)} & \geq \frac{\left( 2a_2 - a_2^2 / (\delta_2^2 + \chi^2_{(k-1; \lambda_0)}) \right) - \delta_1^2 / \sigma^2 + 2\theta_1 \tilde{\theta}_1 - 2a_2 \theta_1 \tilde{\theta}_1}{(\delta_2^2 + \chi^2_{(k-1; \lambda_0)}) + 2a_2 (k-1) / (\delta_2^2 + \chi^2_{(k-1; \lambda_0)})} \\
\chi^2_{(k+1; \lambda_0)} & \geq 4a_1^2 \lambda_0 - (k-1) - 2a_2 (k-1) \\
\chi^2_{(k+1; \lambda_0)} & \geq 8a_2^2 \lambda_0 / (\delta_2^2 + \chi^2_{(k-1; \lambda_0)})^2 \right]\end{align*} \]
\begin{align*}
&\times \left( \chi'_{(k+1;\lambda)}^2 / \chi_{(k+1;\lambda)}^2 \right) + E\left[ I_{(\rho_1, \rho_2)} \left( \hat{\theta}_1 \right) \right] \\
&- I_{(-\infty, a_1 ; \delta ; \sigma)} \left( \hat{\theta}_1 \right) - I_{(\sigma_2 ; \delta)} \left( \hat{\theta}_1 \right) \times I_{(0, a_2 ; \delta ; \sigma)} \left( \hat{\theta}_1 \right) \\
&\times \left( \chi'_{(k+1;\lambda)}^2 \right) \left( 2a_2 - a_2 / \left( \sigma_1^2 / \sigma^2 + \chi_{(k+1;\lambda)}^2 \right) \right) - \sigma_1^2 / \sigma^2 + 2 \theta_1 \hat{\theta}_1 \\
&- 2a_2 \left( \sigma_1^2 / \sigma^2 + \chi_{(k+1;\lambda)}^2 \right) + 2a_2 (k-1) / (\sigma_1^2 / \sigma^2 + \chi_{(k+1;\lambda)}^2) \\
&+ I_{(0, a_2 ; \delta ; \sigma)} \left( \chi_{(k+1;\lambda)}^2 \right) \left( 4a_0 \sigma_1^2 + (k-1) - 2a_2 (k-1) \right) \\
&\left( \sigma_1^2 / \sigma^2 + \chi_{(k+1;\lambda)}^2 \right) - 2a_2 (k-1) / (\sigma_1^2 / \sigma^2 + \chi_{(k+1;\lambda)}^2) \\
&- I_{(0, a_2 ; \delta ; \sigma)} \left( \chi_{(k+3;\lambda)}^2 \right) \left( 2a_0 \lambda_0 / (\sigma_1^2 / \sigma^2 + \chi_{(k+3;\lambda)}^2) \right) \\
&- 8a_2 \lambda_0 / (\sigma_1^2 / \sigma^2 + \chi_{(k+3;\lambda)}^2) \right) \right) ,
\end{align*}

where $\lambda_0 = \theta_0 / (2\sigma^2)$, $\theta_0 = (0, \theta_{(k-1)}')$ and $N(c, j, \tau) = P(\chi_{(j;\tau)}^2 \leq c)$.

Proof: See the Appendix.

From (22), we note that $\rho(\theta^{**}, \theta) \leq \rho(\theta^{**}, \theta)$ when $0 \leq a_1 \leq 2(k-3)$, $0 \leq a_2 \leq 2(k-2)$ and $0 \leq a_3 \leq 2(k-3)$. It can be seen easily from (23) that these are also the conditions for $\rho(\hat{\theta}_1, \theta) \leq \rho(\hat{\theta}_1, \theta)$. Under these conditions, $\hat{\theta}_1$ and $\theta^{**}$ are both inadmissible.

Also, differentiating (22) and (23) shows that the choice of $a_1 = k-3$, $a_2 = k-2$ and $a_3 = k-3$ minimizes the risk of $\theta^{**}$ and $\hat{\theta}_1$. Given that the interval restricted estimator encompasses the inequality restricted estimator, it is not surprising these parameter values are similar to the corresponding values chosen for the optimal Stein inequality restricted estimator given in Judge et al. (1984).

Finally, comparing the risk of $\theta^{**}$ for the optimal values of $a_1$, $a_2$ and $a_3$ with the Stein-rule estimator $\hat{\theta}_1$ for $a = k-2$ shows that

\begin{align*}
\rho(\theta^{**}, \theta) - \rho(\hat{\theta}_1, \theta) &= \rho(\theta^{**}, \theta) - k + E \left[ I_{(-\infty, \rho_1)} \left( \hat{\theta}_1 \right) \right] (k-2)^2 (1/\chi_{(k+1;\lambda)}^2) \\
&- (k-3)^2 (1/\chi_{(k+1;\lambda)}^2) + E \left[ I_{(\rho_2, \delta)} \left( \hat{\theta}_1 \right) \right] (k-2)^2
\end{align*}
INTERVAL RESTRICTED AND PRE-TEST ESTIMATORS

\[ x \left( \frac{1}{\chi^2_{(k; \lambda)}} - (k-3)^2 \frac{1}{\chi^2_{(k-1; \lambda)}} \right) \]  

(26)

From Ohtani (1987), \( \rho(\theta^{**}, \theta) - k \leq 0 \) when the constraint is true or sufficiently true, and \( > 0 \) otherwise. Furthermore, \( \rho(\theta^{**}, \theta) - k \) is independent of \( k \) (see equation (7)). Now, given that \( k-3 < k-2 \), and that \( 1/\chi^2_{(k; \lambda)} \leq 1/\chi^2_{(k-1; \lambda)} \), it is difficult to determine the sign of the expectation terms in (26) for \( k \geq 4 \). Further analytical evaluations of these risk expressions are also difficult. Accordingly, these risk expressions are numerically evaluated and analysed in the next section.

4. NUMERICAL EVALUATIONS OF THE RISK FUNCTIONS

Numerical calculations of (22) – (25) were carried out for \( k = 4, 6, 8; \alpha = 0.025, 0.05; \lambda_0 = 1, 5; d = 0.5, 1, 2, 3; a = a_2 = k-2 \) and \( a_1 = a_3 = k-3 \). The case where \( \rho_1 \to -\infty \) or \( \rho_2 \to \infty \) was also evaluated as a special case. The cumulative Normal distribution function \( F(.) \) and the non-central Chi-Square probabilities \( N(j, c, \tau) \) were calculated using the NAG (1991) subroutine SISABF and Davies' (1980) algorithm respectively. The recurrence formula given in Xie (1988) was used to evaluate 

\[ E\left( \frac{1}{\chi^2_{(j; \lambda_0)}} \right). \]

When \( j \) is an odd integer, the application of Xie's recurrence formula involves the calculation of the Dawson's integral, which can be computed using the NAG subroutine SISFFF. Furthermore, the NAG subroutine DOIFCF was used to evaluate 

\[ E\left( I_{(\cdot)}(\chi^2_{(j; \lambda_0)})/\chi^2_{(j; \lambda_0)} \right). \]

Finally, the expectation terms involving the product of indicator functions \( I_{(\cdot)}(\tilde{\theta}^1_{1} \tilde{\theta}^2_{1} \tilde{\sigma}^2(\chi^2_{(j; \lambda_0)}) \) were evaluated by recursive use of the NAG subroutines DO1AJF and DO1AHF. These were incorporated into a FORTRAN program written by the author and executed on a VAX7610 machine. Figures 1 - 6 illustrate some typical results.

From the diagrams, we observe the following features. First, the interval restricted maximum likelihood estimator \( \theta^{**} \) is uniformly dominated by the Stein restricted estimator \( \theta^{***} \), which is in turn uniformly dominated by its positive-rule variant \( \theta^{*} \). Both \( \theta^{**} \) and
Figure 1. Relative risk functions of $\hat{\theta}^\bullet$, $\hat{\theta}$, $\hat{\theta}^\star$, $\hat{\theta}^\circ$, $\hat{\theta}^*$, $\hat{\theta}^\circ$, $\hat{\theta}^\star$ and $\hat{\theta}^*$ for $\alpha = 0.025$, $k = 4$, $\lambda_0 = 1$ and $d = 0.5$.

Figure 2. Relative risk functions of $\hat{\theta}^\bullet$, $\hat{\theta}$, $\hat{\theta}^\star$, $\hat{\theta}^\circ$, $\hat{\theta}^*$, $\hat{\theta}^\circ$, $\hat{\theta}^\star$ and $\hat{\theta}^*$ for $\alpha = 0.025$, $k = 4$, $\lambda_0 = 1$ and $d = 3$. 

Figure 3. Relative risk functions of \( \theta^{**}, \hat{\theta}, \tilde{\theta}, \tilde{\theta}^{**}, \hat{\theta}_{s}, \tilde{\theta}^{**}_{s}, \tilde{\theta}^{***}_{s}, \) and \( \hat{\theta}^{*} \) for \( \alpha = 0.025, k = 8, \lambda_0 = 1 \) and \( d = 0.5 \).

Figure 4. Relative risk functions of \( \theta^{**}, \hat{\theta}, \tilde{\theta}, \tilde{\theta}^{**}, \hat{\theta}_{s}, \tilde{\theta}^{**}_{s}, \tilde{\theta}^{***}_{s}, \) and \( \hat{\theta}^{*} \) for \( \alpha = 0.025, k = 4, \lambda_0 = 5 \) and \( d = 0.5 \).
Figure 5. Relative risk functions of $\theta^{**}$, $\hat{\theta}$, $\tilde{\theta}$, $\tilde{\theta}^{**}$, $\hat{\theta}^{**}$, $\tilde{\theta}^{**}$, $\tilde{\theta}^{***}$ and $\tilde{\theta}^*$ for $\alpha = 0.05$, $k = 4$, $\lambda_0 = 1$ and $d = 0.5$.

Figure 6. Relative risk functions of $\theta^{**}$, $\hat{\theta}$, $\tilde{\theta}$, $\tilde{\theta}^{**}$, $\hat{\theta}^{**}$, $\tilde{\theta}^{**}$, $\tilde{\theta}^{***}$ and $\tilde{\theta}^*$ for $\alpha = 0.025$, $k = 4$, $\lambda_0 = 1$, $\rho_1 = -\infty$ and $\rho_2 = 1.5$. 
\( \theta^{**} \) are therefore inadmissible. The biggest risk gain of using \( \theta^{**} \) over \( \theta^{**} \) tends to occur at the origin (i.e., the mid-point of the constraint). The same conclusions hold for the corresponding interval pre-test estimators. With an appropriate choice of test size, both the Stein pre-test estimator \( \hat{\theta}^{*} \) and the positive-part Stein pre-test estimator \( \hat{\theta}^{+} \) can strictly dominate \( \theta^{**} \).

In addition, at least for the cases that we have considered, there is always a region where \( \theta^{**} \) and \( \theta^{***} \) are risk superior to their corresponding interval pre-test estimators. This typically occurs when the constraint is correct or nearly so. As \( d \) increases, (i.e., the interval constraint gets wider), the region in which \( \hat{\theta}(\theta^{***}) \) is dominated by \( \theta^{**}(\theta^{***}) \) also widens (compare Figure 1 with Figure 2). However, comparing \( \theta^{**} \) and \( \theta^{***} \) with \( \hat{\theta}^{*} \) and \( \hat{\theta}^{+} \) shows that \( \theta^{**} \) and \( \theta^{***} \) do not necessarily have lower risk than their corresponding unrestricted estimators even when the interval constraint is perfectly correct. Interestingly enough, when \( k \) is relatively large (say, \( > 6 \)), the traditional Stein-rule and positive-part Stein-rule estimators can uniformly dominate their corresponding interval restricted and pre-test estimators (see Figure 3).

Furthermore, as \( \lambda^{0} \) increases, the risk of both \( \tilde{\theta}^{*} \) and \( \tilde{\theta}^{+} \) approach \( \kappa \), the risk of the maximum likelihood estimator \( \tilde{\theta}^{*} \). This has the effect of causing the risk of \( \theta^{**} \), \( \theta^{***} \) and their corresponding pre-test estimators to lie closer to \( \theta^{**} \) and \( \hat{\theta} \) respectively (see Figure 4). Other things being equal, when \( \lambda^{0} \) is large, the risk comparisons among members of the Stein family of estimators that we have considered reveal behaviour similar to that found among maximum likelihood based estimators. This result is not surprising. Recall that the risk of \( \tilde{\theta}^{*} \) and \( \tilde{\theta}^{+} \) approach that of \( \tilde{\theta} \) when \( \lambda \to \infty \). Given that \( \lambda = \lambda^{0} \hat{\theta}^{2}/(2\sigma^{2}) \), when \( \lambda^{0} \) is large, we would expect the risk of \( \tilde{\theta} \) and \( \tilde{\theta}^{*} \) to behave as when \( \lambda \) is large. An analogous argument can be made to explain the risk behaviour of \( \theta^{**} \), \( \theta^{***} \), \( \hat{\theta} \) and \( \hat{\theta}^{*} \) when \( \lambda^{0} \) is large.

Consistent with the results of other pre-test problems that have been analyzed in the literature, a decrease in \( \alpha \) causes the Stein pre-test and positive-part Stein pre-test estimators to lie closer to
their corresponding interval restricted estimators, and vice versa (compare Figures 1 and 5). Finally, when the interval constraint reduces to an inequality constraint, the various interval restricted and pre-test estimators that we have considered become the corresponding inequality restricted and pre-test estimators (see Figure 6). The risk behaviour of these inequality restricted and pre-test estimators is quantitatively different from their corresponding interval restricted and pre-test estimators, but the general conclusions regarding the risk comparison of these estimators are the same as in the case where the restriction is in the form of a fixed interval.

5. CONCLUSIONS

Non-sample prior information often specifies the unknown coefficients in a regression model to lie within an interval constraint. In this paper, we have shown that the positive-part Stein interval restricted estimator is uniformly superior to the other interval restricted estimators that we have considered and thus demonstrated their inadmissibility. The same conclusion applies to the risk comparison of the corresponding interval pre-test estimators. It is also of considerable interest to note that the traditional Stein-rule estimator and its positive-part variant can uniformly dominate their corresponding interval restricted and pre-test estimators. We have shown by numerical evaluation that this occurs when the model has a relatively large number of coefficients and $\lambda_0$ is relatively small. The situation where the prior information is in the form of an inequality constraint is nested as a special case in our analysis.

The risk behaviour of both the Stein interval pre-test estimator and its positive-part variant depends on the choice of $c_1$ and $c_2$ or $\alpha$. The optimal choice of the size of the pre-test is yet to be explored. Finally, the situation where the disturbance variance is unknown has also been analysed via Monte Carlo simulations. In this case, the test statistic $t_i = (\tilde{\theta}_i - \rho_i)/\tilde{\sigma}$, $i = 1, 2$, each have Student's $t$
distribution with n-k degrees of freedom. Although this makes the derivation of the risk functions much more difficult, the qualitative nature of the results is the same as in the $\sigma^2$ known case.

APPENDIX

Proof of Theorem 1:

The derivation of the risk of $\theta^{**}$ follows a similar approach to that given in Judge et al. (1984) for deriving the risk of the inequality Stein restricted estimator. Due to space limitation, we only outline the proof for the derivation of $\rho(\hat{\theta}^*, \theta)$. The expressions for the risk functions of $\hat{\theta}^*$ and $\theta^{***}$ can be derived in a similar fashion, and the derivation is available upon request from the author.

Now, from the definition of risk under squared error loss,

$$\rho(\hat{\theta}^*, \theta) = E\left[ (\hat{\theta}^* - \theta)' (\hat{\theta}^* - \theta) \right] \sigma^2$$  \hspace{1cm} (A1)

Substituting the expression of $\hat{\theta}^*$ given in (21), (A1) can be written as

$$\rho(\hat{\theta}^*, \theta) = \rho(\hat{\theta}^*, \theta) + E\left[ \left( 1 - \left( I_{(-\alpha,c_2)}(u_2) - I_{(-\alpha,c_1)}(u_1) \right) \right) \right]$$

$$\times \left( I_{(0,1)}(\bar{\theta}_0 \bar{\theta}_0) \right) (\sigma^2) (1-a_2 \sigma^2 / \bar{\theta}_0 \bar{\theta}_0)^2 \bar{\theta}_0 \bar{\theta}_0 \sigma^2 \right)$$

$$+ E\left[ \left( I_{(-\alpha,c_2)}(u_2) - I_{(-\alpha,c_1)}(u_1) \right)^2 \right] \left( I_{(-\alpha,\rho_1)}(\tilde{\theta}) \right)$$

$$I_{(0,1)}(\bar{\theta}_0 \bar{\theta}_0) \left( 1-a_1 \sigma^2 / \bar{\theta}_0 \bar{\theta}_0 \right)^2 \bar{\theta}_0 \bar{\theta}_0 \sigma^2 \right)$$

$$+ \left( I_{(\rho_2,\omega)}(\bar{\theta}_0 \bar{\theta}_0) \right) \left( 1-a_3 \sigma^2 / \bar{\theta}_0 \bar{\theta}_0 \right)^2 \bar{\theta}_0 \bar{\theta}_0 \sigma^2 \right)$$

$$- 2E\left[ (\hat{\theta}_0' - \theta)' \left( 1 - \left( I_{(-\alpha,c_2)}(u_2) - I_{(-\alpha,c_1)}(u_1) \right) \right) \right]$$

$$\times \left( I_{(0,1)}(\bar{\theta}_0 \bar{\theta}_0) \right) (\sigma^2) (1-a_2 \sigma^2 / \bar{\theta}_0 \bar{\theta}_0)^2 \bar{\theta}_0 \bar{\theta}_0 \sigma^2 \right)$$

$$- I_{(-\alpha,c_1)}(u_1) \times \left[ \left( I_{(-\alpha,\rho_1)}(\tilde{\theta}) \right) I_{(0,1)}(\bar{\theta}_0 \bar{\theta}_0) \right] (\sigma^2)$$
\[ \begin{align*}
x \cdot (1-a_1^2/\hat{\theta}_0' \hat{\theta}_0) + \left[ I_{(\rho_2, \omega)}(\tilde{\theta}_1) \right]_{(0, a)^2} (\tilde{\theta}_0' \tilde{\theta}_0/\sigma^2) \\
x \cdot (1-a_3^2/\hat{\theta}_0' \hat{\theta}_0) + \left[ I_{(\rho_2, \omega)}(\tilde{\theta}_1) \right]_{(0, a)} (\tilde{\theta}_0' \tilde{\theta}_0/\sigma^2) \\
x \cdot (1-a_3^2/\hat{\theta}_0' \hat{\theta}_0) + \left[ I_{(\rho_2, \omega)}(\tilde{\theta}_1) \right]_{(0, a)} (\tilde{\theta}_0' \tilde{\theta}_0/\sigma^2)
\end{align*} \]

since
\[ \left[ \begin{array}{c} 1 - \left( I_{(-\omega, c_1)}(u_2) - I_{(-\omega, c_1)}(u_1) \right) \\ I_{(-\omega, c_1)}(u_1) \end{array} \right] = 0. \]

As \( u_1 = (\tilde{\theta}_1 - \rho_1)/\sigma \), the indicator functions \( I_{(-\omega, c_1)}(u_1) \) can be written as \( I_{(-\omega, c_1) + \rho_1/\sigma}(\tilde{\theta}_1) \), \( i = 1, 2 \). Furthermore, \( I_{(-\omega, c_1) + \rho_1/\sigma}(\tilde{\theta}_1) = I_{(-\omega, c_1) + \rho_1/\sigma}(\tilde{\theta}_1) \), and \( I_{(-\omega, c_1) + \rho_1/\sigma}(\tilde{\theta}_1) = I_{(-\omega, \rho_1/\sigma)}(\tilde{\theta}_1) \) for \( i = 1, 2 \). Using these results, after some manipulations and recognizing that \( \tilde{\theta}_1 \) is distributed independently of \( \tilde{\theta}_0 \), (A2) reduces to
\[ \rho(\tilde{\theta}_1^*, \theta) = \rho(\tilde{\theta}_1^*, \theta) + E \left[ \left( I_{(-\omega, \rho_1)}(\tilde{\theta}_1) - I_{(-\omega, c_1) + \rho_1/\sigma}(\tilde{\theta}_1) \right) \right] \]
\[ \times E \left[ I_{(0, a)}(\tilde{\theta}_0' \tilde{\theta}_0/\sigma^2) \left( 2a_1^2 \sigma^2 - a_1^2 \sigma^2/\tilde{\theta}_0' \tilde{\theta}_0 - \tilde{\theta}_0' \tilde{\theta}_0 \right) \right] / \sigma^2 + E \left[ \left( I_{(\rho_2, \omega)}(\tilde{\theta}_1) - I_{(\omega, c_2) + \rho_2/\sigma}(\tilde{\theta}_1) \right) \right] \]
\[ + E \left[ I_{(0, a)}(\tilde{\theta}_0' \tilde{\theta}_0/\sigma^2) \left( 2a_3^2 \sigma^2 - a_3^2 \sigma^2/\tilde{\theta}_0' \tilde{\theta}_0 - \tilde{\theta}_0' \tilde{\theta}_0 \right) \right] / \sigma^2 + E \left[ \left( I_{(\rho_2, \omega)}(\tilde{\theta}_1) + I_{(-\omega, c_1) + \rho_1/\sigma}(\tilde{\theta}_1) \right) \right] \]
\[ + E \left[ \left( I_{(\rho_2, \omega)}(\tilde{\theta}_1) + I_{(-\omega, c_1) + \rho_1/\sigma}(\tilde{\theta}_1) \right) \right] \times E \left[ I_{(0, a)}(\tilde{\theta}_0' \tilde{\theta}_0/\sigma^2) \left( 2a_2^2 \sigma^2 - a_2^2 \sigma^2/\tilde{\theta}_0' \tilde{\theta}_0 - \tilde{\theta}_0' \tilde{\theta}_0 \right) \right] / \sigma^2. \quad (A3) \]

Recall that \( \tilde{\theta}_0' \tilde{\theta}_0/\sigma^2 \sim \chi^2_{(k-1; \lambda)} \), and \( \tilde{\theta}_0' \tilde{\theta}_0/\sigma^2 \sim \chi^2_{(k; \lambda)} \). Applying
Theorem 1 and 2 given in Judge and Bock (1978, pp. 321-322).
\[
E \left[ I_{(0,a)} \left( \frac{\bar{\theta}_0^2}{\sigma^2} \bar{\theta}_0 \bar{\theta}_0 \right) / \sigma^2 \right] = E \left[ I_{(0,a)} \left( \chi^2_{(k+1; \lambda_0)} \right) (k-1) \right] + 2\lambda_0 E \left[ I_{(0,a)} \left( \chi^2_{(k+3; \lambda_0)} \right) \right],
\]
for \( i = 1, 3 \).

Furthermore, \( \bar{\theta} \bar{\theta} = \bar{\theta}_1^2 + \bar{\theta}_0 \bar{\theta}_0 \), \( \theta \theta = \theta_1^2 + \theta_0 \theta_0 \) and

\[
E \left[ I_{(0,a)} \left( \bar{\theta} \theta / \sigma^2 \right) \bar{\theta} \theta / \sigma^2 \bar{\theta} \theta / \sigma^2 \right] = E \left[ I_{(0,a)} \left( \chi^2_{(k+1; \lambda_0)} \right) \chi^2_{(k+1; \lambda_0)} \right],
\]
for \( i = 1, 3 \).

Again, using Theorems 1 and 2 of Judge and Bock (1978),

\[
E \left[ I_{(0,a)} \left( \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right) \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right] = E \left[ I_{(0,a)} \left( \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right) \chi^2_{(k+1; \lambda_0)} \right] + 2\lambda_0 E \left[ I_{(0,a)} \left( \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right) \chi^2_{(k+3; \lambda_0)} \right],
\]

and

\[
E \left[ I_{(0,a)} \left( \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right) \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right] = 2\lambda_0 E \left[ I_{(0,a)} \left( \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right) \chi^2_{(k+1; \lambda_0)} \right],
\]

Finally, the expectation term \( E \left[ I_{(0,a)} \left( \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right) \chi^2_{(k+1; \lambda_0)} \right] \) may be evaluated by recursive use of Theorem 7 and Theorem 2 given in Judge and Bock (1978), yielding

\[
E \left[ I_{(0,a)} \left( \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right) \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right] = E \left[ I_{(0,a)} \left( \bar{\theta}_0 \bar{\theta}_0 / \sigma^2 \right) \chi^2_{(k+1; \lambda_0)} \right].
\]
\[ \times \left[ \left( \frac{1}{(0, a \rightarrow 0, \lambda_0)} x_{(k+1; \lambda_0)}^2 / (\tilde{\theta}_1^2 / \sigma^2 + x_{(k+1; \lambda_0)}^2) \right) (k-1) 
+ \left( \frac{1}{(0, a \rightarrow 0, \lambda_0)} x_{(k+3; \lambda_0)}^2 / (\tilde{\theta}_1^2 / \sigma^2 + x_{(k+3; \lambda_0)}^2) \right) 2 \lambda_0 
- \left( \frac{1}{(0, a \rightarrow 0, \lambda_0)} x_{(k+1; \lambda_0)}^2 / (\tilde{\theta}_1^2 / \sigma^2 + x_{(k+1; \lambda_0)}^2) \right) (k-1) 
+ \left( \frac{1}{(0, a \rightarrow 0, \lambda_0)} x_{(k+3; \lambda_0)}^2 / (\tilde{\theta}_1^2 / \sigma^2 + x_{(k+3; \lambda_0)}^2) \right) 2(k-1) 
+ \left( \frac{1}{(0, a \rightarrow 0, \lambda_0)} x_{(k+1; \lambda_0)}^2 / (\tilde{\theta}_1^2 / \sigma^2 + x_{(k+1; \lambda_0)}^2) \right) 4 \lambda_0 \right] . \]

(A10)

Substituting these results into (A3) and rewriting \( E \left( I_{(1)} \tilde{\theta}_1 \right) \) and \( E \left( I_{(1)} \tilde{\theta}_0 \tilde{\theta}_0 / \sigma^2 \right) \) as cumulative density functions for the Normal and non-central Chi-Square distributions yield (25) directly.

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